Math Calculus Review

CHSN Review Project

Contents

This review guide was written by Dara Adib. Portions of the "Limits" and "Derivatives" chapters are based off the Calculus Wikibook available on the Internet at `http://en.wikibooks.org/wiki/Calculus`. CHSN Review Project contributors Dara Adib and Paul Sieradzki contributed to the "Limits" section of the Calculus Wikibook.

This is a development version of the text that should be considered a work-in-progress.

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Limits

This chapter was originally designed for a test on limits administered by Jeanine Lennon to her Math 12H (4H/Precalculus) class on April 2, 2008. It was later updated with an "Addendum" section (page 12) for a test on limits administered by Jonathan Chernick to his AP[1] Calculus BC class on September 18, 2008.

Introduction

A limit looks at what happens to a function when the input approaches, but does not necessarily reach, a certain value. The general notation for a limit is below.

$$\lim_{x \to c} f(x) = L$$

This is read as "the limit of $f(x)$ as x approaches c is L."

Informal Definition of a Limit

L is the limit of $f(x)$ as x approaches c. The value of $f(x)$ comes close to L when x is close (but not necessarily equal) to c. It can be represented by either of the following forms, with the former being far more common.

- $\lim\limits_{x \to c} f(x) = L$
- $f(x) \to L$ as $x \to c$

Rules

Now that a limit has been informally defined, some rules that are useful for manipulating a limit are listed.

Identities

The following identities assume $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$. Using these identities, other rules can be deduced.

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Scalar Multiplication

A scalar is a constant. When a function is multiplied by a constant, scalar multiplication is performed.

$$\lim_{x \to c} kf(x) = k \cdot \lim_{x \to c} f(x) = kL$$

Addition

$$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$$

Subtraction

$$\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$$

Multiplication

$$\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = L \cdot M$$

Division

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}, \text{ where } M \neq 0$$

Constant Rule

The constant rule states that if $f(x) = k$ is constant for all x, then the limit as x approaches c must be equal to k.

$$\lim_{x \to c} k = k$$

Identity Rule

The identity rule states that if $f(x) = x$, then the limit as x approaches c is equal to c.

$$\lim_{x \to c} x = c$$

Power Rule

The rule for products many times results in determining the power rule.

$$\lim_{x\to c} f(x)^n = \left(\lim_{x\to c} f(x)\right)^n$$

Finding Limits

If c is in the domain of the function and the function can be built out of rational, trigonometric, logarithmic and exponential functions, then the limit is simply the value of the function at c.

If c is not in the domain of the function, then in many cases (as with rational functions) the domain of the function includes all of the points near c, but not c. An example would be if one wanted to find $\lim_{x\to 0} \frac{x}{x}$, where the domain includes all real numbers except 0. In that case, one would want to find a similar function, with the hole filled in. The limit of this function at c will be the same, while the function is the same at all points not equal to c. The limit definition depends on $f(x)$ only at the points where x is close to c but not equal to it. And since the domain of the new function includes c, one can now (assuming it's still built out of rational, trigonometric, logarithmic and exponential functions) just evaluate the function at c as before.

In the above example, this is easy; canceling the x's gives 1, which equals x/x at all points except 0. Thus, $\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$. In general, when computing limits of rational functions, it's a good idea to look for common factors in the numerator and denominator.

Does Not Exist

Note that the limit might not exist at all. There are a number of ways in which this can occur.

Not Same from Both Sides

A left-handed limit is different from the right-handed limit of the same variable, value, and function. Since, the left-handed limit ≠ right-handed limit, the limit does not exist. This includes cases in which the limit of a certain side does not exist (e.g. $\lim_{x\to 2} \sqrt{x-2}$, which has no left-handed limit).

Gap

There is a gap (more than a point wide) in the function where the function is not defined. As an example, in $f(x) = \sqrt{x^2 - 16}$, $f(x)$ does not have any limit when $-4 \leq x \leq 4$. There is no way to "approach" the middle of the graph. Note also that the function also has no limit at the endpoints of the two curves generated (at $x = -4$ and $x = 4$) since limits from both sides do not exist.

Jump

If the graph suddenly jumps to a different level, there is no limit. This is illustrated in the floor function (in which the output value is the greatest integer not greater than the input value). The limit does not exist when the greatest integer function approaches an integer ($\lim_{x \to integer} \lfloor x \rfloor$, also written as int x). $|x|/x$ and $x/|x|$ are other examples of graphs that contain jumps.

Infinite Oscillation

This can be tricky to visualize. A graph continually rises above and below a horizontal line as it approaches a certain x-value, for instance infinity. This often means that the limit does not exist, as the graph never approaches a particular value. However, if the height (and depth) of each oscillation diminishes as the graph approaches the x-value, so that the oscillations get arbitrarily smaller, then there might actually be a limit.

The use of oscillation naturally calls to mind the trigonometric functions. An example of a trigonometric function that does not have a limit as x approaches 0 is $f(x) = \sin \frac{1}{x}$. As x gets closer to 0, the function keeps oscillating between -1 and 1.

Incomplete Graph

Consider the following example.

$$g(x) = \begin{cases} 2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

$g(x)$ does not have a limit. For let x be a real number, $g(x)$ can't have a limit at x. No matter how close one gets to x, there will be rational numbers (when $g(x)$ will be 2) and irrational numbers (when g will be 0). Thus $g(x)$ has no limit at any real number.

One-Sided Limits

Sometimes, it is necessary to consider what happens when one approaches an x value from one particular direction. To accommodate for this, there are one-sided limits. In a left-handed limit, x approaches a from the left hand side (negative). Likewise, in a right-handed limit, x approaches a from the right hand side (positive).

For example, $\lim_{x \to 2} \sqrt{x-2}$ does not exist because there is no left-handed limit.

The left-handed limit, which does not exist, is expressed as the following.

$$\lim_{x \to 2^-} \sqrt{x-2}$$

The right-handed limit, which equals 0, is expressed as the following.

$$\lim_{x \to 2^+} \sqrt{x-2} = 0$$

Infinite Limits

Limits can also involve looking at what happens to $f(x)$ as x gets very big. For example, consider the function $f(x) = \frac{1}{x}$. As x becomes very big, $\frac{1}{x}$ becomes closer to zero. Without limits it is very difficult to talk about this fact, because $\frac{1}{x}$ never actually becomes zero. But the language of limits exists precisely to let one talk about the behavior of a function as it approaches something, without caring about the fact that it will never get there. In this case, however, the same problem as before exists; how big does x have to be to be sure that $f(x)$ is really going towards 0?

In this case, the bigger x gets, the closer $f(x)$ should get to 0. Really, this means that however close one wants $f(x)$ to get to 0, one can find an x big enough so $f(x)$ is that close. This is written in a similar way to finite limits and is read as "the limit, as x approaches infinity, equals 0," or "as x approaches infinity, the function approaches 0."

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Rules

The easiest way to determine limits as x approaches $\pm\infty$ is by using the graphing calculator to make observations, or by plugging in high values of positive and negative numbers in a calculator.

However, there are three rules for determining a limit of a fraction analytically as a variable approaches infinity. For each rule, one must look at the variables on both the numerator and denominator of the function.

Look for the highest term (with the highest exponent) in the numerator. Look for the same in the denominator. These rules are based on that information.

For limits as the variable approaches infinity:

- If the exponent of the highest term in the numerator matches the exponent of the highest term in the denominator, the limit is the fractional ratio of the coefficients of the highest terms.
- If the *numerator* has the highest term, then the fraction is called "top heavy" and the limit is infinity.
- If the *denominator* has the highest term, then the fraction is called "bottom heavy" and the limit is zero.

If there is no denominator stated, it is understood that the denominator is 1 or $1n^0$, and the limit will be infinity.

Asymptotes

A linear asymptote is a straight line that a graph approaches, but does not become identical to. Asymptotes are formally defined using limits.

Vertical Asymptotes

The line $x = a$ is a vertical asymptote for the function $f(x)$ if at least one of the following statements is true.

1. $\lim_{x \to a} f(x) = \pm\infty$
2. $\lim_{x \to a^-} f(x) = \pm\infty$
3. $\lim_{x \to a^+} f(x) = \pm\infty$

The limits from both directions do not have to be equal to have an asymptote, but they may be equal. Essentially, a vertical asymptote occurs where the the value of a limit is positive or negative infinity from any direction.

Recall that this occurs where the fraction of a function is undefined (denominator equals zero).

Removable Discontinuities

The function $f(x) = \frac{x^2-9}{x-3}$ is considered to have a removable discontinuity at $x = 3$. It is discontinuous at that point because the fraction then becomes $\frac{0}{0}$ which is undefined.

Standard algebraic techniques for simplifying fractions and algebraic expressions (i.e. factoring, multiplying by conjugates) can be used to eliminate the discontinuity.

$$f(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{(x-3)} = \frac{x+3}{1} \cdot \frac{x-3}{x-3} = \frac{x+3}{1} \cdot 1 = x+3$$

However, the function is not really continuous, and an open circle must be left in the graph at the removable discontinuity.

Horizontal Asymptotes

The line $y = a$ is a horizontal asymptote for the function $f(x)$ if $\lim_{x \to \infty} f(x) = a$ or $\lim_{x \to -\infty} f(x) = a$.

If $\lim_{x \to \infty} f(x) = a$ and $\lim_{x \to -\infty} f(x) = b$, then the function $f(x)$ has two asymptotes at $y = a$ and $y = b$. Note that in some functions, the graph may pass through the horizontal asymptote at an x value of zero.

Essentially, a horizontal asymptote occurs at the value of a limit where x approaches positive or negative infinity.

Recall that rules exist for calculating the the value of a limit where x approaches positive or negative infinity.

Rules

The easiest way to determine limits as x approaches $\pm\infty$ is by using the graphing calculator to make observations, or by plugging in high values of positive and negative numbers in a calculator.

However, there are three rules for determining a limit of a fraction analytically as a variable approaches infinity. For each rule, one must look at the variables on both the numerator and denominator of the function.

Look for the highest term (with the highest exponent) in the numerator. Look for the same in the denominator. These rules are based on that information.

- If the exponent of the highest term in the numerator matches the exponent of the highest term in the denominator, the limit is the fractional ratio of the coefficients of the highest terms.
- If the *numerator* has the highest term, then the fraction is called "top heavy" and the limit is infinity.
- If the *denominator* has the highest term, then the fraction is called "bottom heavy" and the limit is zero.

If there is no denominator stated, it is understood that the denominator is 1 or $1n^0$, and the limit will be infinity.

Sketching with Asymptotes

A series of steps can be taken to sketch with asymptotes. As a result, curves may be sketched without a graphing calculator.

1. Find the x-intercept by setting y equal to zero.
2. Find the y-intercept by setting x equal to zero.
3. Find the horizontal asymptote(s).
4. Find the vertical asymptotes(s).
5. Plot the x-intercept and y-intercept.
6. Sketch the asymptote(s).
7. Find the limits of both sides of the vertical asymptote by using test points.
8. Sketch the curve using the determined information and the sketched asymptotes.

In some problems only limits will be provided. From these limits horizontal and vertical asymptotes can be determined. While the x-intercept and y-intercept are not provided, it is still possible to sketch the graph. The sketch will be less accurate, but that is acceptable when provided with *limit*ed information.

Continuity

Definition

The formal definition of continuity is simple.

If $f(x)$ is defined on an open interval containing c, then $f(x)$ is said to be continuous at c if and only if the limit as x approaches c equals $f(c)$.

$$\lim_{x \to c} f(x) = f(c)$$

Note that for $f(x)$ to be continuous at c, the definition requires three conditions.

1. $f(x)$ is defined at c
   a) $f(c)$ exists
2. The limit as x approaches c exists.
   a) $\lim_{x \to c} f(x)$ exists
3. The limit and $f(c)$ are equal.
   a) $f(c) = \lim_{x \to c} f(x)$

If any of these do not hold then $f(x)$ is not continuous at c.

Notice how this relates to the idea of continuity. To be continuous, the function must be uniformly "smooth" (e.g. no "gaps," breaks, or sharp turns/corners) within an interval.

A function is said to be continuous if it is continuous at every point c in its domain.

A function may be continuous at a certain point, but not a continuous function (throughout). Likewise, a discontinuous function may be continuous at a certain point.

Removable Discontinuities

discontinuity point where a function is not continuous

If there is a "gap" one point wide on a graph ($f(c)$ does not exist) or if there is a "jump" one point wide on a graph ($f(c) \neq \lim_{x \to c} f(x)$), the discontinuity is removable. Gap discontinuities ($\lim_{x \to c} f(x)$ does not exist), jump discontinuities ($f(c) \neq \lim_{x \to c} f(x)$), and infinite oscillation discontinuities are non-removable.

The function $f(x) = \frac{x^2-9}{x-3}$ is considered to have a removable discontinuity at $x = 3$. It is discontinuous at that point because the fraction then becomes $\frac{0}{0}$ which is undefined. Therefore the function fails the very first condition of continuity.

If the function is slightly modified, the discontinuity can be removed and the function becomes continuous. Standard algebraic techniques for simplifying fractions and algebraic expressions (e.g. factoring, multiplying by conjugates) can be used.

To make the function $f(x)$ continuous, $f(x)$ must be simplified.

$$f(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{(x-3)} = \frac{x+3}{1} \times \frac{x-3}{x-3} = \frac{x+3}{1} \times 1 = x+3$$

As long as $x \neq 3$, the function $f(x)$ can be simplified to get a new function $g(x)$.

$$g(x) = \begin{cases} x+3, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases}$$

Note that the function $g(x)$ is not the same as the original function $f(x)$, because $g(x)$ has the extra point $(3, 6)$. $g(x)$ is now defined for $x = 3$, and therefore continuous.

Properties

If $f(x)$ and $g(x)$ are continuous, then the following are also continuous:

- $f(x) + g(x)$
- $f(x) \cdot g(x)$
- $f(x) - g(x)$
- $\frac{f(x)}{g(x)}$, $g \neq 0$
- $k \times f(x)$, where k is a constant

Intermediate Value Theorem

A graph of a continuous function has no breaks, so a point between two x-values has a y-value between the y-values of the respective x-values.

If a function is continuous on the closed interval $[a, b]$, then for every value k between $f(a)$ and $f(b)$ there is a value c on $[a, b]$ such that $f(c) = k$.

This can be used to approximate when the y-value of a function will be a certain value (e.g. the x-value when $y = 4$).

Calculating Continuities

One should be able to determine where a function is discontinuous. In some cases, one may be required to determine the value(s) of variable(s) in rule(s) of a function so that the function will be continuous. A system of equations is required when there are multiple variables.

Trigonometric Functions

In most cases, limits with trigonometric functions can be treated the same way as other limits.

One can substitute into the expression if possible, or use the graphing calculator.

If divide by zero occurs, one may eliminate removable discontinuities if they exist or use the graphing calculator. In some cases, factoring to eliminate removable discontinuities can only be done if trigonometric identities are used first.

Note When graphing, stay in radian mode as the limits are provided in radian mode unless stated otherwise.

Trigonometric Identities

Trigonometric identities (page 42) can be used to simplify expressions before or after finding a limit.

Addendum

This section was designed for a test on limits administered by Jonathan Chernick to his AP[2] Calculus BC class on September 18, 2008. It is not covered in Math 12H/4H.

Further Trigonometric Identities

These identities can be used for the same purpose as the other trigonometric identities. To use these identities, the limits may need to be multiplied by a certain factor or separated based on the rules on page 3.

Sine

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Cosine

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Squeeze (Sandwich) Theorem

The squeeze theorem, also known as the sandwich theorem, is used to find the limit of a function by comparison with two other functions whose limits are known or easily computed. It refers to a function $f(x)$ whose values are squeezed between the values of two other functions $g(x)$ and $h(x)$, both of which have the same limit L. If the value of $f(x)$ is trapped between the values of the two functions $g(x)$ and $h(x)$, the values of $f(x)$ must also approach L.

If the following are true:

1. $g(x) \leq f(x) \leq h(x)$ for all x not equal to c
2. $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

Then $\lim_{x \to c} f(x) = L$.

Example:

$$\lim_{x \to 0} x \sin \frac{1}{x}$$

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Note that the sine of anything is in the interval $[-1, 1]$. In other words, $-1 \leq \sin x \leq 1$ for all x). As a result, for all nonzero x, $-1 \times |x| \leq x \sin \frac{1}{x} \leq 1 \times |x|$. Simplified, this means $-|x| \leq x \sin \dfrac{1}{x} \leq |x|$. Since $\lim\limits_{x \to 0} -|x| = \lim\limits_{x \to 0} |x| = 0$, $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$.

End Behavior

The end behavior of a graph describes how it appears as x approaches infinity to the right (x increases) or to the left (x decreases). End behavior is expressed as a behavior model. The behavior model of a graph depends on the highest order term in the equation. In rational expressions (fractions), this would be the division of the highest order term in the numerator by the highest order term in the denominator.

For example, the behavior model of $\dfrac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$ is $\dfrac{2x^5}{3x^2}$. The limit as x approaches both positive and negative infinity would be positive infinity.

Differing Behavior

Sometimes, right-hand and left-hand behavior differ.

If the function is $f(x)$ and its left-hand behavior model is $g(x)$, $\lim\limits_{x \to \infty^-} \dfrac{f(x)}{g(x)} = 1$. Likewise, if the function is $f(x)$ and its right-hand behavior model is $h(x)$, $\lim\limits_{x \to \infty^+} \dfrac{f(x)}{h(x)} = 1$.

Example:

$$f(x) = x + e^{-x}$$

$\lim\limits_{x \to \infty^-} \dfrac{x + e^{-x}}{e^{-x}} = \lim\limits_{x \to \infty^-} \dfrac{x}{e^{-x}} + \lim\limits_{x \to \infty^-} \dfrac{e^{-x}}{e^{-x}} = 0 + 1 = 1$. Therefore, $y = e^{-x}$ is the left-hand behavior model.

$\lim\limits_{x \to \infty^+} \dfrac{x + e^{-x}}{x} = \lim\limits_{x \to \infty^+} \dfrac{x}{x} = \lim\limits_{x \to \infty^+} \dfrac{e^{-x}}{x} = 1 + 0 = 1$. Therefore, $y = x$ is the right-hand behavior model.

Derivatives

This chapter was originally designed for a test on derivatives administered by Jeanine Lennon to her Math 12H (4H/Precalculus) class on April 18, 2008. It was updated for a quiz on the derivatives of trigonometric functions on April 29, 2008, and later updated with an "Addendum" section (page 27) for a test on derivatives administered by Jonathan Chernick to his AP[3] Calculus BC class on October 14, 2008.

Introduction

The slope of a curve cannot be determined by using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, but the slopes of tangent lines drawn to a curve can be determined. To create an infinite number of tangent lines, two points on the curve must be "pushed" together so that their distance, h, approaches zero.

The concept of a limit is used to find a derivative. The derivative is the m_{tan} (slope of tangent line) on a curve at a specific point.

derivative slope of a curve at a given point on the curve

normal line line perpendicular to a tangent line at the point of tangency

Definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Tangent Lines

The derivative can be used to calculate the equation of a line tangent to a curve at a certain point. The derivative is the slope of the tangent line, and when the coordinates of the certain point on the curve are known, the calculated slope and the coordinates of the certain point on the curve (values can be calculated by plugging into equation of curve) can be plugged into $y = mx + b$ or the point-slope formula to determine the equation of the tangent line.

If the slope of a curve at a given point (derivative) is equal to the slope of another curve at a given point, then the two curves have parallel tangent lines at the indicated points.

Notation

The derivative notation is special and unique in mathematics. There are two kinds of notations: Leibniz notation and Newtonian notation.

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Leibniz Notation

The Leibniz notation is expressed as $\frac{dy}{dx}$, meaning "rate of change in y with respect to x" or as $\frac{d}{dx}$, which literally means "derivative with respect to x." Because the derivative of function y is defined as a function representing the slope of function y, the second (or double) derivative is the function representing the slope of the first derivative function. In Leibniz notation, this is written as:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Newtonian Notation

With the Newtonian notation, the derivative of the function $f(x)$ is denoted by $f'(x)$, and its second (or double) derivative is denoted by $f''(x)$. This is read as "f double prime of x," or "the second derivative of $f(x)$."

Higher Order Derivatives

The second derivative is the derivative of the derivative of a function. Subsequent derivatives can be calculated by calculating the derivative of the previous derivative. The following are notations for derivatives of different orders.

Order	Newtonian Notation	Leibniz Notation	Leibniz Notation
First Derivative	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$
Second Derivative	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$
Third Derivative	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$
Fourth Derivative	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$
N^{th} Derivative	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$

One should not write $f^n(x)$ to indicate the n^{th} derivative, as this is easily confused with the quantity $f(x)$ all raised to the n^{th} power.

Rules

Rules for calculating the derivatives of general functions have been developed. As a result, it is possible to calculate the derivative of a wide variety of functions. In many cases the use of multiple rules are required.

Constant Function

For any constant c,

$$\frac{d}{dx}[c] = 0$$

The function $f(x) = c$ is a horizontal line, which has a constant slope of zero. Therefore, it should be expected that the derivative of this function is zero, regardless of the value of x. It is important to understand that e and π are constants, and that their derivative is therefore zero.

Linear Function

For any constants m and c,

$$\frac{d}{dx}[mx + c] = m$$

The function $f(x) = mx + c$ is a line with a slope of m.

Constant Multiplier Rule

For any constant c,

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

In the definition of a derivative, one can factor c out of the numerator and then out of the entire limit.

Addition/Subtraction Rule

For the given functions $f(x)$ and $g(x)$,

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

As a result, one can take an equation, break it up into terms, figure out the derivative individually, and build the answer back up.

Power Rule

For any constant exponent n,

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}, x \neq 0$$

This rule is actually in effect in linear equations too, since $x^{n-1} = x^0$ when $n = 1$, and any real number or variable to the zero power is one.

This rule also applies to fractional and negative powers. Therefore,

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Since polynomials are sums of monomials, using this rule and the addition/subtraction rule (page 16) lets one calculate the derivative of any polynomial.

Simple Fractions

When taking the derivative of simple fractions, one can use the following shortcut to quickly do so. The calculations of derivatives of more complex fractions require use of the quotient rule.

$$\frac{d}{dx}\left[\frac{c}{x^b}\right] = \frac{d}{dx}\left[cx^{-b}\right] = -cbx^{-b-1} = -cbx^{-(b+1)} = \frac{-cb}{x^{b+1}}, \text{ where c is a constant}$$

Chain Rule

The chain rule allows one to calculate the derivative of an unexpanded expression without expanding the expression. This is done by calculating the derivative of the composite of two functions.

For example, see the function $f(x) = (a+b)^c$. To make this the composite of two functions, $g(x) = a + b$ and $f(x) = g(x)^c$. This function can be rewritten as the composite function $f(g(x))$, where $g(x)$ is the polynomial $(a + b)$ and $f(x)$ is $g(x)$ to the c^{th} power.

According to the chain rule,

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \times g'(x)$$

An example of this situation is $f(x) = (3x+4)^3$. In this case, $g(x) = 3x+4$ and $f(x) = g(x)^3$. According to the chain rule,

$$\frac{d}{dx}\left[(3x+4)^3\right] = 3(3x+4)^2 \times \frac{d}{dx}[3x+4] = 3(3x+4)^2 \times (3+0) = 9(3x+4)^2$$

Product Rule

The derivative of the function $f(x) = A \times B$ would *not* be $f'(a) \times f'(b)$. The product rule allows one to correctly calculate the derivative of the product of two functions.

According to the product rule,

$$\frac{d}{dx}[f(x) \times g(x)] = f(x) \times g'(x) + g(x) \times f'(x)$$

The derivative of the product of two functions is the first function multiplied by the derivative of the other function, added to the first function multiplied by the derivative of the second function.

The mnemonic device "one-D-two plus two-D-one" can be used to remember this rule.

Quotient Rule

As with multiplying, the derivative of a quotient is not the quotient of the derivatives. The quotient rule allows one to correctly calculate the derivative of the quotient of two functions.

According to the quotient rule,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \times f'(x) - f(x) \times g'(x)}{g(x)^2}$$

The mnemonic device "low-D-high minus high-D-low over the square of what's below" can be used to remember this rule.

Basic Polynomials

With these rules, the derivative of any polynomial can be determined. Here is a step-by-step example of the process of calculating the derivative of a fairly simple polynomial. The chain, product, and quotient rules are not covered.

$\frac{d}{dx}\left[6x^5 + 3x^2 + 3x + 1\right]$

The addition/subtraction rule (page 16) splits the equation into several terms.

$\frac{d}{dx}\left[6x^5\right] + \frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}[3x] + \frac{d}{dx}[1]$

The constant (page 16) and linear (page 16) rules get rid of some terms.

$\frac{d}{dx}\left[6x^5\right] + \frac{d}{dx}\left[3x^2\right] + 3 + 0$

The constant multiplier rule (page 16) moves the constants outside of the derivatives.

$6\frac{d}{dx}\left[x^5\right] + 3\frac{d}{dx}[x] + 3$

The power rule (page 16) works on the individual monomials.

$6\left(5x^4\right) + 3(2x) + 3$

Simplifying obtains the final answer.

$30x^4 + 6x + 3$

Graphing Calculator

In some cases it may be easier or required to calculate derivatives using the graphing calculator. It can also be used to check one's answer.

There are two methods of calculating a derivative of a graph with a Texas Instruments graphing calculator. These instructions are designed for a TI-84 Plus calculator, but they may used on other Texas Instruments graphing calculators, though slight modification may be necessary.

Unless otherwise specified, the graphing calculator should be in radian mode.

1. Math $\longrightarrow$ 8 (8. nDeriv) $\longrightarrow$ enter with form *function*,x,x *value* $\longrightarrow$ Enter
   a) replace *function* with the appropriate function
   b) replace x *value* with the appropriate value
2. Graph function $\longrightarrow$ 2nd $\longrightarrow$ Trace (Calc) $\longrightarrow$ enter x value $\longrightarrow$ Enter
   a) use the appropriate x value

Does Not Exist

The graphing calculator may display an incorrect answer when calculating derivatives that do not exist (e.g. at a corner). Graphing calculators like the TI-84 Plus calculate derivatives by using the symmetric difference quotient.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

The problem with this method is that the calculator will actually calculate the average slope over a very small area instead of the true derivative (instantaneous slope). At a corner, the average slope over a very small area will be zero, but the correct answer is that the derivative does not exist.

Differentiability

Definition

For $f(x)$ to be differentiable at point c, the following must be true:

1. $f(x)$ must be continuous at point c
   a) $f(x)$ is defined at c
      i. $f(c)$ exists
   b) The limit as x approaches c exists.
      i. $\lim_{x \to c} f(x)$ exists
   c) The limit and $f(c)$ are equal.
      i. $f(c) = \lim_{x \to c} f(x)$
2. The derivative from both sides must be equal
   a) $\lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$

If any of these do not hold then $f(x)$ is not differentiable at c.

Notice how this relates to the idea of differentiability. To be differentiable, the function must have a uniform rate of change (e.g. no corners, cusps, or vertical tangents) within an interval.

A function is said to be differentiable if it is differentiable at every point c in its domain.

A function may be differentiable at a certain point, but not a differentiable function (throughout). Likewise, a non-differentiable function may be differentiable at a certain point.

Not Differentiable

Corner

A function does not have a derivative at a corner.

$$\lim_{x \to a^-} f'(x) \neq \lim_{x \to a^+} f'(x)$$

Cusp

A cusp occurs when the limit of the slope from one side of a curve goes to $-\infty$ and the other side of the curve goes to $+\infty$. As a result, a function does not have a derivative at a cusp.

$$\lim_{x \to a^-} f'(x) \neq \lim_{x \to a^+} f'(x)$$

Vertical Tangent

A function does not have a derivative at a vertical tangent.

$$\lim_{x \to a} f'(x) = \infty$$

Endpoint

A function is not differentiable at an endpoint because the derivative can only be calculated from one side. However, since an endpoint has a one-sided derivative, the endpoints on the graph of the derivative of a function are filled in.

> Endpoints are a source of a lot of seeming inconsistency in calculus.

Trigonometric Functions

Trigonometric Identities

Trigonometric identities (page 42) can be used to simplify expressions before or after finding a derivative.

Derivation

Sine, cosine, tangent, cotangent, secant, and cosecant are trigonometric functions. Each trigonometric function has a derivative.

Trigonometric Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \times \tan x$
$\csc x$	$-\csc x \times \cot x$

Sine

The derivative of sine is cosine.

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

Cosine

The derivative of cosine is negative sine.

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

Tangent

Using the quotient rule (page 18) and the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$, the derivative of tangent can be derived.

$$\begin{aligned} \tan(x) &= \frac{\sin(x)}{\cos(x)} \\ \frac{d}{dx}[\tan(x)] &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ \frac{d}{dx}[\tan(x)] &= \frac{1}{\cos^2(x)} \\ \frac{d}{dx}[\tan(x)] &= \sec^2(x) \end{aligned}$$

Therefore, the derivative of tangent is the square of secant.

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

Cotangent

Using the quotient rule (page 18) and the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$, the derivative of cotangent can be derived.

$$\begin{aligned}
\cot(x) &= \frac{\cos(x)}{\sin(x)} \\
\frac{d}{dx}[\cot(x)] &= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \\
\frac{d}{dx}[\cot(x)] &= \frac{-1}{\sin^2(x)} \\
\frac{d}{dx}[\cot(x)] &= -\csc^2(x)
\end{aligned}$$

Therefore, the derivative of cotangent is the negative of the square of cosecant.

$$\boxed{\frac{d}{dx}[\cot(x)] = -\csc^2(x)}$$

Secant

Using the quotient rule (page 18), the derivative of secant can be derived.

$$\begin{aligned}
\sec(x) &= \frac{1}{\cos(x)} \\
\frac{d}{dx}[\sec(x)] &= \frac{\sin(x)}{\cos^2(x)} \\
\frac{d}{dx}[\sec(x)] &= \frac{1}{\cos(x)} \times \frac{\sin(x)}{\cos(x)} \\
\frac{d}{dx}[\sec(x)] &= \sec(x) \times \tan(x)
\end{aligned}$$

Therefore, the derivative of secant is secant multiplied by tangent.

$$\boxed{\frac{d}{dx}[\sec(x)] = \sec(x) \times \tan(x)}$$

Cosecant

Using the quotient rule (page 18), the derivative of cosecant can be derived.

$$\begin{aligned}
\csc(x) &= \frac{1}{-\sin(x)} \\
\frac{d}{dx}[\csc(x)] &= -\frac{\cos(x)}{\sin^2(x)} \\
\frac{d}{dx}[\csc(x)] &= -\frac{1}{\sin(x)} \times \frac{\cos(x)}{\sin(x)} \\
\frac{d}{dx}[\csc(x)] &= -\csc(x) \times \cot(x)
\end{aligned}$$

Therefore, the derivative of cosecant is the negative of cosecant multiplied by cotangent.

$$\frac{d}{dx}[\csc(x)] = -\csc(x) \times \cot(x)$$

Combining with Derivative Rules

In most cases, one must determine the derivative of an an example that requires the use of derivative rules in addition to the knowledge of the derivatives of trigonometric function. One may apply the form trig (a) to many examples, where trig is the trigonometric function and a is the angle.

Based on the chain rule (page 17), the derivative of trig (a) would be $(\frac{d}{dx}[\text{trig}])(a) \times \frac{d}{dx}[a]$.

$$\frac{d}{dx}[\text{trig}\,(a)] = \left(\frac{d}{dx}[\text{trig}](a)\right) \times \frac{d}{dx}[a],$$

where $\frac{d}{dx}[\text{trig}]$ is the derivative of the trigonometric function, and $\frac{d}{dx}[a]$ is the derivative of the angle.

Example:

$$\sin(2x)$$

$$\begin{aligned}
\frac{d}{dx}[\sin(2x)] &= \left(\frac{d}{dx}[\sin]\,(2x)\right) \times \frac{d}{dx}[2x] \\
\frac{d}{dx}[\sin(2x)] &= (\cos(2x)) \times 2 \\
\frac{d}{dx}[\sin(2x)] &= 2\cos(2x)
\end{aligned}$$

Asymptotes

A linear asymptote is a straight line that a graph approaches, but does not become identical to. Asymptotes are formally defined using limits. See the the asymptotes section of the limits chapter on page 7 for more information.

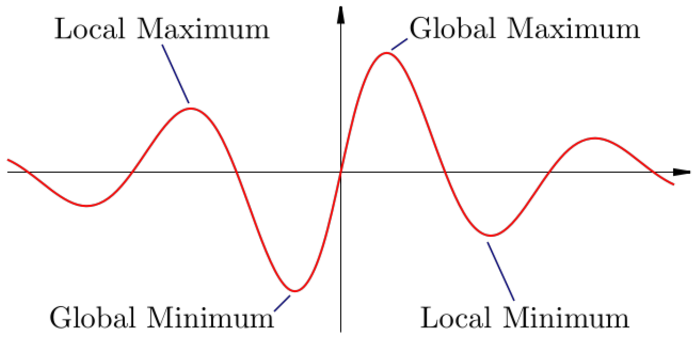

Figure 0.1: Graph demonstrating extrema on a curve

Stationary Points

Extrema

Maxima and minima are points where a function reaches a highest or lowest value, respectively. A maximum occurs when positive slope changes to negative slope and a minimum occurs when negative slope changes to a positive slope. There are two kinds of extrema (a word meaning maximum or minimum): global and local, sometimes referred to as "absolute" and "relative," respectively. A global maximum is a point that takes the largest value on the entire range of the function, while a global minimum is the point that takes the smallest value on the range of the function. Local extrema are the largest or smallest values of the function in the immediate vicinity. See Figure 0.1.

All extrema look like the crest of a hill or the bottom of a bowl on a graph of the function. A global extremum is always a local extremum too, because it is the largest or smallest value on the entire range of the function, and therefore also in its vicinity. It is also possible to have a function with no extrema, global or local (e.g. $y = x$).

At an extremum, the y-value is the value of the extremum and the x-value is where the extremum occurs.

"Flatpoints"

It is important to note that not all cases in which the first derivative of a function is equal to zero are turning points or extrema, though the first derivative of a function is equal to zero or does not exist at all turning points and extrema. "Flatpoints" (e.g. triple roots) may also occur when the first derivative of a function is equal to zero, but they are not turning points nor extrema because no slope change occurs.

Classification

At any extremum, the slope of the graph is zero or undefined, as the graph must stop rising or falling at an extremum, and begin to fall or rise. Because of this, extrema are also commonly called stationary points or turning points. If the graph has one or more of these stationary points, these may be found by setting the first derivative equal to zero and finding the roots of the resulting equation as well as values where the function is undefined. These values are referred to as critical points. Note that if the domain is restricted, the endpoints of the domain must also be checked to see if they are global extrema.

critical point point in domain of f where $f' = 0$ or f' does not exist

Extrema can only occur at critical points and endpoints.

True extrema require a sign change in the first derivative. This makes sense — the graph must rise (positive first derivative) and fall (negative first derivative) to form a maximum. In between rising and falling, on a smooth curve, there will ideally be a point of zero slope — the maximum. A minimum would exhibit similar properties, but in reverse.

First Derivative Test

This leads to a simple method to classify a stationary point — plug x values (test points) slightly left and right into the derivative of the function. If the results have opposite signs then it is a true extremum. To calculate the coordinates of the minimum or maximum point, one would plug the determined x value into the original function to find its y value.

- If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $f(c)$ is a local minimum.
- If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $f(c)$ is a local maximum.

Caution must be exercised with this method, as, if a point too far from the extremum is picked, one could take it on the far side of another extremum and incorrectly classify the point. A more rigorous method to classify a stationary point is called the extremum test that uses the second derivative, but this simple method is acceptable.

Second Derivative Test

- If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum.
- If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum.

Note that the second derivative test cannot be used to verify an extrema if the first or second derivative does not exist.

Information

Stationary Point	First Derivative	Second Derivative
Minimum Point	zero or undefined	positive or undefined
Maximum Point	zero or undefined	negative or undefined
"Flatpoint"	zero	zero

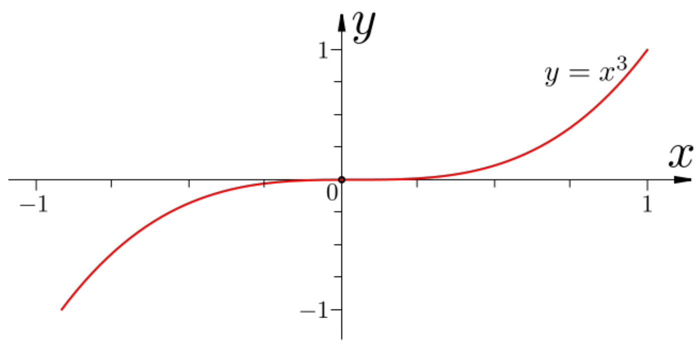

Figure 0.2: Graph containing an inflection point

Stationary Point	First Derivative Sign Before	First Derivative Sign After
Minimum Point	negative	positive
Maximum Point	positive	negative
"Flatpoint"	same sign	same sign

Inflection Points

Inflection points occur when the second derivative of a function is equal to zero. The curve changes from being concave up (positive second derivative) to concave down (negative second derivative), or vice versa. See Figure 0.2. "Flatpoints" are a specific type of inflection point where the graph flattens out (first derivative is zero), but the sign of the slope does not change. These points are called stationary points of inflection. Other inflection points are not "flatpoints," and there is no flattening out (i.e. sine curve); these points are known as non-stationary points of inflection.

Curvature	Second Derivative	First Derivative Graph
Concave Up ("smile")	positive	increasing
Concave Down ("frown")	negative	decreasing
Inflection Point	zero or undefined	extrema

Optimization

Optimization is the use of Calculus in the real world. Calculus is a useful tool for maximizing or minimizing (also known as "optimizing") a situation.

Formulas

Volume

cube $A = a^3$, where a is the length of the side of each edge of the cube

rectangular prism $V = abc$, where a, b, and c are the lengths of the 3 sides of the prism
cylinder $V = \pi r^2 h$, where r is the radius and h is the height of the cylinder
sphere $V = \frac{4}{3}\pi r^3$, where r represents the radius of the sphere

Surface Area

cube $A = 6a^2$, where a is the length of the side of each edge of the cube
rectangular prism $A = 2ab + 2bc + 2ac$, where a, b, and c are the lengths of the 3 sides of the prism
sphere $A = 4\pi r^2$, where r is radius of the sphere
cylinder $A = 2\pi r^2 + 2\pi rh$, where r is the radius and h is the height of the cylinder

Addendum

This section was designed for a test on derivatives administered by Jonathan Chernick to his AP[4] Calculus BC class on October 14, 2008. It is not covered in Math 12H/4H.

Alternative Definition of Derivative

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Parametric Equations

Parametric equations are typically defined by two equations that specify both the x and y coordinates of a graph using a parameter. They are graphed using the parameter (usually t) to figure out both the x and y coordinates.

The derivative of the parametrized curve $x(t), y(t)$ is:
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \frac{dx}{dt} \neq 0$

Example:

$$x = t, y = t^2$$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{1} = 2t$

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Implicit Differentiation

explicit relationship function in which $f(x)$ is given in terms of x and constants; for every x-value there is one y-value

implicit relationship relationship between two or more variables; two or more functions put together

Ordinary differentiation is explicit differentiation. Implicit differentiation is useful when differentiating an equation that cannot be explicitly differentiated because it is impossible or hard to isolate variables (e.g. $x^2 + xy + y^2 = 16$).

In many difficult problems involving implicit differentiation (e.g. multiple choice), it is necessary to substitute the dependent variable (e.g. y) and its derivatives (e.g. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$) based on the original equation or previous determined derivative expressions.

Example:

$$x^2 + y^2 = 1$$

Explicit Differentiation

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= 1 - x^2 \\ y &= \pm\sqrt{1 - x^2} \\ y &= \pm(1 - x^2)^{\frac{1}{2}} \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Implicit Differentiation

$$\begin{aligned} x^2 + y^2 &= 1 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

Inverse Functions

inverse function "opposite" of a function; if $f(x) = a$, $f^{-1}(a) = f(x)$; reflected over line $y = x$

The composition of a function and its inverse is x because the two functions "undo" each other.

$$f\left(f^{-1}(x)\right) = x$$

With use of the chain rule (page 17), the relationship between the derivative of a function and the derivative of its inverse can be determined.

$$\begin{aligned} f\left(f^{-1}(x)\right) &= x \\ f'\left[f^{-1}(a)\right] \times \left[f^{-1}\right]'(a) &= 1 \\ \left[f^{-1}\right]'(a) &= \frac{1}{f'\left[f^{-1}(a)\right]} \end{aligned}$$

A function and its inverse have reciprocal slopes with reversed (x, y) values.

$$\left[f^{-1}\right]'(a) = \frac{1}{f'\left[f^{-1}(a)\right]}$$

Example: $f(x) = x^3 + x - 2$, find $\left[f^{-1}\right]'(0)$

$$\begin{aligned} 0 &= x^3 + x - 2 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} f'(x) &= 3x^2 + 1 \\ f'(1) &= 4 \end{aligned}$$

$$\begin{aligned} \left[f^{-1}\right]'(a) &= \frac{1}{f'\left[f^{-1}(a)\right]} \\ \left[f^{-1}\right]'(0) &= \frac{1}{f'\left[f^{-1}(0)\right]} \\ \left[f^{-1}\right]'(0) &= \frac{1}{f'(1)} \\ \left[f^{-1}\right]'(0) &= \frac{1}{4} \end{aligned}$$

Inverse Trigonometric Functions

The inverse trigonometric functions are the inverse functions of the trigonometric functions. The inverse of the trigonometric functions sin, cos, tan, cot, sec, and csc is arcsin, arccos, arctan, arccot, arcsec, and arccsc, respectively.

The notations $\sin^{-1}$, $\cos^{-1}$, etc. are often used for arcsin, arccos, etc., respectively, but this convention may result in confusion between multiplicative inverse and compositional inverse since this logically conflicts with the structure of expressions like $\sin^2 x$, which do not refer to function composition but rather multiplication.

Each inverse trigonometric function has a derivative.

Trigonometric Function	Inverse (arc notation)	Inverse ($^{-1}$ notation)
sin	arcsin	$\sin^{-1}$
cos	arccos	$\cos^{-1}$
tan	arctan	$\tan^{-1}$
cot	arccot	$\cot^{-1}$
sec	arcsec	$\sec^{-1}$
csc	arccsc	$\csc^{-1}$

In the table below, u can represent any differentiable expression, using the chain rule (page 17).

Inverse Trigonometric Function	Derivative
$\arcsin u$	$\frac{1}{\sqrt{1-u^2}} \times \frac{du}{dx}, \lvert u \rvert < 1$
$\arccos u$	$\frac{-1}{\sqrt{1-u^2}} \times \frac{du}{dx}, \lvert u \rvert < 1$
$\arctan u$	$\frac{1}{1+u^2} \times \frac{du}{dx}$
$\text{arccot}\, u$	$\frac{-1}{1+u^2} \times \frac{du}{dx}$
$\text{arcsec}\, u$	$\frac{1}{\lvert u \rvert \sqrt{u^2-1}} \times \frac{du}{dx}, \lvert u \rvert > 1$
$\text{arccsc}\, u$	$\frac{-1}{\lvert u \rvert \sqrt{u^2-1}} \times \frac{du}{dx}, \lvert u \rvert > 1$

Strategies for Simplifying In many difficult problems (e.g. multiple choice) where simplifying is necessary, there are some strategies for doing so. If simplifying is not required, these strategies are not necessary.

- If an expression under an absolute value is always positive, the absolute value symbols can be removed.
- Combine terms into terms with a common denominator.
- Factor out variables from square roots.

More Rules

If the original expression is a constant raised to a variable power, use the c^x rule (). If the original expression contains a variable in the base and exponent, logarithmic differentiation (page 31) must be used.

e^x

The derivative of e^x is itself.

Based on the chain rule (page 17), where u is any differentiable expression,
$\frac{d}{dx}[e^u] = e^u \times \frac{du}{dx}$

c^x

c represents a constant. The derivative of c^x is $c^x \times \ln c$, $c > 0$ and $c \neq 1$.

Based on the chain rule (page 17), where c is a constant,
$\frac{d}{dx}[c^u] = \ln c \times c^u \times \frac{du}{dx}$, $c > 0$ and $c \neq 1$

$\ln x$

The derivative of $\ln x$ is $\frac{1}{x}$, $x > 0$.

Based on the chain rule (page 17), where u is any differentiable expression,
$\frac{d}{dx}[\ln u] = \frac{1}{u} \times \frac{du}{dx}$, $u > 0$

Logarithms

Properties These properties hold true for both log and ln.

- $\log(xy) = \log x + \log y$
- $\log(x/y) = \log x - \log y$
- $\log x^a = a \ln x$

Change of Base $\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$

$\log_b x$ The derivative of $\log_b x$ is $\frac{1}{x \ln(b)}$.

Based on the chain rule (page 17), where u is any differentiable expression,
$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln(b)} \times \frac{du}{dx}$; $b > 0$, $b \neq 1$, and $u > 0$

Logarithmic Differentiation Logarithmic differentiation is a differentiation process used to take the derivative of a variable raised to a variable or other complex situations. The natural log (ln) of both sides of an equation are taken, and the result is implicitly differentiated.

Extreme Value Theorem

If f is continuous on the interval $[a, b]$, f has both a maximum and a minimum value in the interval.

Note that brackets [] refer to a closed interval including the endpoints while parentheses ( ) refer to an interval not including the endpoints.

Mean Value Theorem

If f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b), there exists a point c on (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

In other words, somewhere on the interval the slope of the tangent line equals (at least once) the slope of the secant line connecting the two endpoints.

Rolle's Theorem

Rolle's Theorem is a special case of the Mean Value Theorem.

If f is continuous on the interval $[a, b]$, differentiable on the interval (a, b), and $f(a) = f(b)$, then there exists a point c on (a, b) such that $f'(c) = 0$.

Integrals

This chapter was designed for a test on integrals administered by Jonathan Chernick to his AP[5] Calculus BC class on November 26, 2008. It is not covered in Math 12H/4H.

Definite Integrals

Definition

definite integral area between a curve and the x-axis (area underneath the x-axis is negative)

A finite number of rectangles can be used to estimate this area. A larger number of rectangles will give a more accurate estimate, and an infinite number of rectangles can give an exact answer.

$$\int_a^b [f(x)dx] \approx A_k = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

Riemann Sums

This area can be expressed as the infinite limit of Riemann sums. As n gets larger the width of the rectangles gets smaller and when n approaches infinity, the exact area is calculated.

If $f(x)$ is a continuous on the closed interval $[a, b]$, the definite integral of $f(x)$ between a and b is:

$$\int_a^b [f(x)]dx = \lim_{n\to\infty} \left(\sum_{k=1}^{n} f(c_k) \right) \left(\frac{b-a}{n} \right)$$

where c_k are sample points in the interval.

Notation

When considering the expression $\int_a^b [f(x)]dx$, the function $f(x)$ is called the integrand and the interval $[a, b]$ is the interval of integration. a and b are the lower and upper limits of integration, respectively.

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Rectangular Approximation Method

Rectangular Approximation Method (RAM) is a method of estimating definite integrals by calculating the area of a certain number of rectangles. A larger number of rectangles will give a more accurate estimate.

Left Rectangular Approximation Method (LRAM)

$$\int_a^b [f(x)dx] \approx \Delta x(f(a) + f(a + \Delta x) + \cdots + f(b - 2\Delta x) + f(b - \Delta x))$$

where Δx is the width of the rectangles ($\frac{b-a}{n}$) and n is the number of rectangles.

Right Rectangular Approximation Method (RRAM)

$$\int_a^b [f(x)dx] \approx \Delta x(f(a + \Delta x) + f(a + 2\Delta x) + \cdots + f(b - \Delta x) + f(b))$$

where Δx is the width of the rectangles ($\frac{b-a}{n}$) and n is the number of rectangles.

Midpoint Rectangular Approximation Method (MRAM)

$$\int_a^b [f(x)dx] \approx \Delta x(f(a + \frac{\Delta x}{2}) + f(a + \Delta x) + \cdots + f(b - \Delta x) + f(b - \frac{\Delta x}{2}))$$

where Δx is the width of the rectangles ($\frac{b-a}{n}$) and n is the number of rectangles.

Trapezoidal Approximation Method

$$\int_a^b [f(x)dx] \approx \left(\frac{1}{2}\right)(\Delta x)\left(f(a) + 2f(a + \Delta x) + \cdots + 2f(b - \Delta x) + f(b)\right)$$

where Δx is the width of the trapezoids ($\frac{b-a}{n}$) and n is the number of trapezoids.

An integral approximated with this rule on a concave-up function will be an overestimate because the trapezoids include all of the area under the curve and extend over it. Using this method on a concave-down function yields an underestimate because area is unaccounted for under the curve, but none is counted above.

Graphing Calculator

These instructions are designed for a TI-84 Plus calculator, but they may used on other Texas Instruments graphing calculators, though slight modification may be necessary. Unless otherwise specified, the graphing calculator should be in radian mode.

Definite Integral Rectangular Approximations

In some cases it may be easier or required to calculate rectangular approximations of definite integrals using the graphing calculator, especially when using a large number of rectangles.

The program `RAM` must be added to the calculator's memory. Once installed, set the y_1 of the calculator's graph to the function being integrated and run the program with `PRGM` $\longrightarrow$ `RAM`.

Definite Integral Calculations

In some cases it may be easier or required to calculate definite integrals using the graphing calculator, especially when the function is too complex. It can also be used to check one's answer.

Fundamental Theorem of Calculus

Every continuous function has an antiderivative.

Part I

If f is continuous on the closed interval $[a, b]$ and $F(x) = \int_a^x [f(t)dt]$ on the closed interval $[a, b]$, then F is differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all x in the open interval (a, b).

By definition $F(x)$ is the antiderivative of $f(x)$ in the open interval (a, b).

Part II

If f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f, then:

$$\int_a^b [f(x)dx] = F(b) - F(a)$$

It is therefore possible to calculate a definite integral using rules for antiderivatives (indefinite integrals).

Corollary

Integration and differentiation are inverses of each other.

If f is continuous on the closed interval $[a, b]$ then:

$$\frac{d}{dx}\left[\int_a^x [f(t)dt]\right] = f(x)$$

$$\frac{d}{du}\left[\int_a^u [f(t)\ dt]\right] = f(u)du$$

Integral Rules

Rules for calculating the integrals of general functions have been developed. As a result, it is possible to calculate the integrals of a wide variety of functions. In many cases the use of multiple rules are required.

In the following rules, C represents the constant of integration.

Constant Function

The definite integral of a constant function is a rectangle with the height being the constant and the width being the interval of integration.

$$\int [c\,dx] = cx + C$$

$$\int_a^b [c\,dx] = c(b - a)$$

where c is a constant.

Addition/Subtraction Rule

If $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$, then:

$$\int [(f(x) \pm g(x))dx] = \int [f(x)dx] \pm \int [g(x)dx] + C$$

$$\int_a^b [(f(x) \pm g(x))dx] = \int_a^b [f(x)dx] \pm \int_a^b [g(x)dx]$$

As a result, one can take an equation, break it up into terms, figure out the definite integrals individually, and build the answer back up.

Constant Multiplier Rule

$$\int [c \times f(x) dx] = c \int [f(x) dx]$$

$$\int_a^b [c \times f(x) dx] = c \int_a^b [f(x) dx]$$

Power Rule

$$\int [x^n dx] = \frac{x^{n+1}}{n+1} + C$$

$$\int_a^b [x^n dx] = \frac{b^{n+1} - a^{n+1}}{n+1}$$

where n is a constant exponent not equal to -1 and $x \neq 0$.

Expressions containing roots (i.e. square roots) can be intregrated by using a fractional value for n ($\sqrt[b]{x^a} = x^{a/b}$). Expressions containing algebraic monomials in the denominator of a fraction can be integrated by inverting the sign of n ($\frac{1}{x^n} = x^{-n}$).

Logarithms

$\frac{1}{x}$ Rule

$$\int \left[\frac{dx}{x}\right] = \ln|x| + C$$

$$\int_a^b \left[\frac{dx}{x}\right] = \ln|b| - \ln|a|$$

where $x \neq 0$.

e^x Rule

$$\int \left[e^{kx} dx\right] = \frac{e^{kx}}{k} + C$$

$$\int_a^b \left[e^{kx} dx\right] = \frac{e^{kb}}{k} - \frac{e^{ka}}{k}$$

where k is a constant.

a^x Rule

$$\int [a^x dx] = \frac{a^x}{\ln a} + C$$

Trigonometry

- integrating the derivatives of the six trigonometric functions
- integrating the derivatives of the inverse trigonometric functions

See the the trigonometric section of the derivatives chapter on page 20 for more information.

Constant

If the constant is outside the trigonometric function, use the constant multiplier rule (Section). If the constant is inside the trigonometric function, use the following rule.

$$\int [(\text{trig}\, kx)dx] = \frac{(\int [\text{trig}]kx)}{k} + C$$

where k is a constant.

Definite Integrals

Additivity Rule

The area under the graph of $f(x)$ between a and b is the area between a and c plus the area between c and b.

$$\int_a^b [f(x)dx] = \int_a^c [f(x)dx] + \int_c^b [f(x)dx]$$

Zero Rule

$$\int_a^a [f(x)dx] = 0$$

Order of Integration Rule

$$\int_b^a [f(x)dx] = -\int_a^b [f(x)dx]$$

Mean Value of Definite Integrals

Mean Value

The average (arithmetic mean) y-value of a function over an interval is the integral over the interval divided by the length of the interval.

$$f_{avg} = \frac{\int_a^b [f(x)dx]}{b-a}$$

Mean Value Theorem

If f is continuous on the closed interval $[a, b]$, then at some point c in $[a, b]$ there exists the following:

$$f(c) = \frac{\int_a^b [f(x)dx]}{b-a}$$

Initial Value Problems

Introduction

An equation that contains a derivative is called a differential equation. For example, $\frac{dy}{dx} = 2x$ is a differential equation. Every differential equation of a function corresponds to a specific equation at a particular point (referred to as a particular solution), assuming the point is in the function's domain.

An initial value problem provides a differential equation and a particular point through which the function passes through. The specific equation is determined by calculating the value of C.

Example

$$\frac{dy}{dx} = 2x, \quad y(1) = 6$$

$$\begin{aligned}
\int \left[\frac{dy}{dx}\right] &= \int [2x dx] \\
y &= x^2 + C \\
6 &= (1)^2 + C \\
6 &= 1 + C \\
C &= 5 \\
y &= x^2 + 5
\end{aligned}$$

Slope Fields

Slope fields (also known as direction fields) are a logical extension to initial value problems as they provide a sketch of the differential equation for any value of C.

A table containing the value of $\frac{dy}{dx}$ (the function's slope) at different x and y values is used to create a slope field.

Approaches

These approaches reduce the time required to make or analyze slope fields and the possibility of making errors.

Patterns

Horizontal Pattern When the differential equation only contains the letter y (e.g. $\frac{dy}{dx} = y$), there is a horizontal pattern.

Vertical Pattern When the differential equation only contains the letter x (e.g. $\frac{dy}{dx} = x$), there is a vertical pattern.

Direction of Slope

Determining whether the slopes of points in a certain vicinity are positive or negative is useful for comparing slope fields.

Zero/No Slope

Determining where the slopes of points are infinity (vertical and undefined) and where they are zero is useful for comparing slope fields.

Separation of Variables

Separation of variables is one method to isolate variables in a differentiable equation. The separated variables can then be integrated.

If $\frac{dy}{dx} = g(x)h(y)$, then $\frac{dy}{h(y)} = g(x)dx$.

Basically, $\frac{dy}{dx}$ is being treated as a fraction, which can be can be separated.

Example

$$
\begin{aligned}
\frac{dy}{dx} &= yx \\
\frac{dy}{y} &= x dx \\
\ln|y| &= \frac{1}{2}x^2 \\
e^{\ln|y|} &= e^{\frac{1}{2}x^2} \\
y &= e^{\frac{1}{2}x^2+C}
\end{aligned}
$$

Integration By Substitution

Integration by substitution is a method for integrating a composition of function, when the entire integral can be expressed in terms of constants, u, and du.

Integration by substitution may be used in combination with rules for inverse trigonometric functions.

Trigonometric Identities

Trigonometric identities (page 42) can be used to simplify expressions before or after integrating.

Appendix

Trigonometric Identities

Pythagorean Identities

1. $$\sin^2\theta + \cos^2\theta = 1$$
2. $$1 + \tan^2\theta = \sec^2\theta$$
3. $$1 + \cot^2\theta = \csc^2\theta$$

Quotient Identities

1. $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
2. $$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Sum of Two Angles

1. $$\sin(A + B) = \sin A\cos B + \cos A\sin B$$
2. $$\cos(A + B) = \cos A\cos B - \sin A\sin B$$