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Financial Economics
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Economics 136: Financial Economics
Section Notes for Week 1

1 Concepts

1.1 Probability and Statistics Review

1.1.1 Random Variables

In finance we typically model the return of a financial asset as a random variable. A random variable can be thought of as a listing of possible numerical values as well as the probability that those values will occur.

Formally, a random variable is a function the maps possible states of the world into real numbers. The probabilities express the likelihood that each of those states will occur. For example, imagine that over the next year four possible things can happen with the following probabilities: there is a 40% chance that both oil prices and GDP increase, there is a 20% chance that oil prices decrease while GDP increases, there is a 30% chance that oil prices increase while GDP decreases, and finally there is a 10% chance that both oil prices and GDP decrease. The following table labels each possible state (s_1 - s_4) and shows its probability.

s_1 = Oil up, GDP up	$\Pr(s_1) = 0.40$
s_2 = Oil down, GDP up	$\Pr(s_2) = 0.20$
s_3 = Oil up, GDP down	$\Pr(s_3) = 0.30$
s_4 = Oil down, GDP down	$\Pr(s_4) = 0.10$

Let's consider the stock of an oil company whose profits depend mainly upon oil prices, but also to some degree on the pace of the economy which drives demand for oil. Imagine that the net simple return of the stock is 30% if s_1 occurs, -10% if s_2 occurs, 20% if s_3 occurs, and -15% if s_4 occurs. Notice that the net simple return of the oil stock is a random variable. It is a function which takes a possible future state of the world as an input and returns a real number. Let $X(s_i)$ denote the net simple return of the oil stock (where i can equal 1, 2, 3, or 4).

s_1 = Oil up, GDP up	$\Pr(s_1) = 0.40$	$X(s_1) = 0.30$
s_2 = Oil down, GDP up	$\Pr(s_2) = 0.20$	$X(s_2) = -0.10$
s_3 = Oil up, GDP down	$\Pr(s_3) = 0.30$	$X(s_3) = 0.20$
s_4 = Oil down, GDP down	$\Pr(s_4) = 0.10$	$X(s_4) = -0.15$

As a shorthand, we usually drop the arguement of the function and simply write X, knowing that since it is a random variable, it's value implicitly depends upon which state of the world is realized next year.

1.1.2 Probability Distribtuion Function

For discrete distributions such as in the one in the example above, the probability distribution function is a function that takes any possible value that the random variable can assume as an input and returns the probability of that value occurring. In the example above the random variable X can assume the values 0.30, -0.10, 0.20, or -0.15. We will denote the probability distribution function as f_X. From the table above we can see that there is a probability of 0.4 that the net simple return of the oil stock is 0.30. Thus the we would write $f_X(0.30) = 0.40$

$s_1 =$ Oil up, GDP up	$\Pr(s_1) = 0.40$	$X(s_1) = 0.30$	$f_X(0.30) = 0.40$
$s_2 =$ Oil down, GDP up	$\Pr(s_2) = 0.20$	$X(s_2) = -0.10$	$f_X(-0.10) = 0.20$
$s_3 =$ Oil up, GDP down	$\Pr(s_3) = 0.30$	$X(s_3) = 0.20$	$f_X(0.20) = 0.30$
$s_4 =$ Oil down, GDP down	$\Pr(s_4) = 0.10$	$X(s_4) = -0.15$	$f_X(-0.15) = 0.10$

Note that that the outputs of the probability distribution function sum to one ($0.40 + 0.20 + 0.30 + 0.10 = 1$).

1.1.3 Expected Value

The expected value of a random variable can be thought of as a probability-weighted average of the possible values that the random variable can assume (the "center of mass" of the probability distribution function). For a discrete random variable X, the expected value is defined as,

$$E[X] = \sum_S X f_X(x)$$

Where S denotes the set of possible future states (in this case $S = \{s_1, s_2, s_3, s_4\}$). This seems confusing, but remember that it is just shorthand for,

$$E[X(s_i)] = \sum_{i=1}^{N} X(s_i) f_X(X(s_i)) = \sum_{i=1}^{N} X(s_i) \Pr(X = X(s_i)) = \sum_{i=1}^{N} X(s_i) \Pr(s_i)$$

where N is the total number of possible future states.

Let's find the expected value of the net simple return of the oil stock,

$$\begin{aligned} E[X] &= \sum_{i=1}^{N} X(s_i) \Pr(s_i) \\ &= X(s_1) \Pr(X = X(s_1)) + X(s_2) \Pr(X = X(s_2)) \\ &\quad + X(s_3) \Pr(X = X(s_3)) + X(s_4) \Pr(X = X(s_4)) \\ &= 0.30 * 0.40 + (-0.10) * 0.20 + 0.20 * 0.30 + (-0.15) * 0.10 \\ &= 0.145 = 14.5\% \end{aligned}$$

Note: expectations are linear. So for random variables X, Y, and constants a, b,

$$E[aX + bY] = E[aX] + E[bY] = aE[X] + bE[Y]$$

1.1.4 Variance

The variance of a random variable is a measure of the degree of spread of the probability distribution function around the expected value of the random variable. For a discrete random variable X, the variance is defined as,

$$Var[X] = E[(X - E[X])^2] = \sum_S (X - E[X])^2 f_X(x)$$

Now we can find the variance of the oil stock,

$$\begin{aligned} Var[X] &= \sum_{i=1}^{N} (X(s_i) - E[X])^2 \Pr(s_i) \\ &= (X(s_1) - E[X])^2 \Pr(s_1) + (X(s_2) - E[X])^2 \Pr(s_2) \\ &\quad +(X(s_3) - E[X])^2 \Pr(s_3) + (X(s_4) - E[X])^2 \Pr(s_4) \\ &= (0.30 - 0.145)^2 * 0.40 + (-0.10 - 0.145)^2 * 0.20 \\ &\quad +(0.20 - 0.145)^2 * 0.30 + (-0.15 - 0.145)^2 * 0.10 \\ &= 0.031225 \end{aligned}$$

The standard deviation is defined as the square root of the variance. In this case,

$$\sigma(X) = 0.1767 = 17.7\%$$

Note: an alternate expression for variance can be found with a little algebra (and remembering that $E[X]$ is a constant and can be pulled out of an expectation by linearity),

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] \\ &= E[X^2] - 2E[XE[X]] + E[(E[X])^2)] \\ &= E[X^2] - 2E[X]E[X] + (E[X])^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

Note: the linearity of expectations and some algebra can be used to show how multiplying or adding a constant to a random variable affects its variance. Assume X is a random variable and a and b are constants.

$$\begin{aligned} Var[aX + b] &= E[(aX + b - E[aX + b])^2] \\ &= E[(aX + b - aE[X] + b)^2] \\ &= E[(aX - aE[X])^2] \\ &= E[(a(X - E[X]))^2] \\ &= E[a^2(X - E[X])^2] \\ &= a^2 E[(X - E[X])^2] \\ &= a^2 Var[X] \end{aligned}$$

1.1.5 Covariance

The covariance of two random variables X and Y is defined as,

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Let's extend our original example by adding another stock. Let's assume that Y is a random variable equal to the net simple return on the stock of a car manufacturer. Here are how its returns depend upon the future state of the world. I also include its probability distribution function.

$s_1 =$ Oil up, GDP up	$\Pr(s_1) = 0.40$	$Y(s_1) = 0.30$	$f_Y(0.30) = 0.40$
$s_2 =$ Oil down, GDP up	$\Pr(s_2) = 0.20$	$Y(s_2) = 0.50$	$f_Y(0.50) = 0.20$
$s_3 =$ Oil up, GDP down	$\Pr(s_3) = 0.30$	$Y(s_3) = -0.30$	$f_Y(-0.30) = 0.30$
$s_4 =$ Oil down, GDP down	$\Pr(s_4) = 0.10$	$Y(s_4) = -0.20$	$f_Y(-0.20) = 0.10$

Applying the formulas for expectation, variance, and standard deviation we get $E[Y] = 0.11$, $Var[Y] = 0.1049$, $\sigma[Y] = 0.324$. Looking at the net simple returns of both stocks together,

$s_1 =$ Oil up, GDP up	$\Pr(s_1) = 0.40$	$X(s_1) = 0.30$	$Y(s_1) = 0.30$
$s_2 =$ Oil down, GDP up	$\Pr(s_2) = 0.20$	$X(s_2) = -0.10$	$Y(s_2) = 0.50$
$s_3 =$ Oil up, GDP down	$\Pr(s_3) = 0.30$	$X(s_3) = 0.20$	$Y(s_3) = -0.30$
$s_4 =$ Oil down, GDP down	$\Pr(s_4) = 0.10$	$X(s_4) = -0.15$	$Y(s_4) = -0.20$

$$\begin{aligned}
Cov(X,Y) &= \sum_{i=1}^{N} \Pr(s_i)(X(s_i) - E[X])(Y(s_i) - E[Y]) \\
&= 0.40(0.30 - 0.145)(0.30 - 0.11) \\
&\quad +0.20(-0.10 - 0.145)(0.50 - 0.11) \\
&\quad +0.30(0.20 - 0.145)(-0.30 - 0.11) \\
&\quad +0.10(-0.15 - 0.145)(-0.20 - 0.11) \\
&= -0.00495
\end{aligned}$$

Note:

$$\begin{aligned}
Cov(aX, bY) &= E[(aX - E[aX])(bY - E[bY])] \\
&= E[a(X - E[X])b(Y - E[Y])] \\
&= a \cdot b \cdot E[(X - E[X])(Y - E[Y])] \\
&= a \cdot b \cdot Cov(X,Y)
\end{aligned}$$

Note:

$$\begin{aligned}
Var(aX + bY) &= E[(aX + bY - E[aX + bY])^2] \\
&= E[(aX + bY - aE[X] - bE[Y])^2] \\
&= E[(aX - aE[X] + bY - bE[Y])^2] \\
&= E[(a(X - E[X]) + b(Y - E[Y]))^2] \\
&= E[a^2(X - E[X])^2 + b^2(Y - E[Y])^2 + 2a(X - E[X])b(Y - E[Y])] \\
&= a^2E[(X - E[X])^2] + b^2E[(Y - E[Y])^2] + 2 \cdot a \cdot b \cdot E[(X - E[X])b(Y - E[Y])] \\
&= a^2Var[X] + b^2Var[Y] + 2 \cdot a \cdot b \cdot Cov[X,Y]
\end{aligned}$$

1.1.6 Correlation

While the sign of a covariance is informative, the magnitude of a covariance is hard to interpret. Correlation is a scaled version of covariance. To calculate the correlation of two random variables it is necessary to scale the covariance by the product of the standard deviations of the random variables. The correlation between two random variables can range from -1 to 1. When the correlation equals 1 we say that the random variables are perfectly correlated, and when the correlation equals -1 we say that the random variables are perfectly negatively correlated.

$$Corr(X,Y) = \rho(X,Y) = \frac{Cov(X,Y)}{\sigma(X)\sigma(Y)}$$

Let's calculate the correlation between the net simple returns of the oil stock and the car manufacturer stock,

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma(X)\sigma(Y)} = \frac{-0.00495}{0.1767 * 0.324} = -0.08649$$

1.1.7 Continuous Random Variables

Most of the definitions and derivations above either apply directly to contiuous random variables or can be applied to continuous random variables by replacing summation with integration. One exception to note is that the probability of a continuous random variable taking on any specific value is zero. However the probability that a continuous random variable is in a specific interval can be found by integrating the probability density function over that interval.