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Financial Economics
Economics 136
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Economics 136: Financial Economics
Section Notes for Week 6

1 Bond Duration

1.1 Concepts

Macaulay's duration, D, is defined as

$$D = \frac{-\frac{dP}{P}}{\frac{d(1+R)}{1+R}} = \frac{-d\log P}{d\log(1+R)}$$

where R denotes the yield to maturity of the bond.

Note, the second equality follows because

$$\begin{aligned} \frac{d(\log x)}{dx} &= \frac{1}{x} \\ d(\log x) &= \frac{dx}{x} \end{aligned}$$

Macaulay's duration is sometimes referred to as the **effective maturity** of a bond.

For a zero-coupon bond $P = F/(1+R)^T$ thus

$$\begin{aligned} \log P &= \log F - T\log(1+R) \\ D &= \frac{-d\log P}{d\log(1+R)} \\ &= \frac{-d(\log F - T\log(1+R))}{d\log(1+R)} \\ &= \frac{-d\log F}{d\log(1+R)} + \frac{Td\log(1+R)}{d\log(1+R)} \\ &= 0 + T \end{aligned}$$

which shows that the duration of a zero coupon is equal to its maturity. The duration of a coupon bond is more complicated, but can be thought of as the center of mass the present value of all the bond's payments. In other words, duration is like weighted average maturity (with each payment time-weighted by the size of the present value of that payment).

D is always a decreasing function of the bond's coupon rate ($\frac{\partial D}{\partial(coup.rate)} < 0$).

D is always a decreasing function of the yield to maturity ($\frac{\partial D}{\partial(1+R)} < 0$).

D generally increases with a bond's maturity, but there are cases when D decreases with a bond's maturity.

Modified duration, D^*, is defined as,

$$D^* = \frac{D}{1+R} = \frac{-\frac{dP}{P}}{d(1+R)}$$

1.2 Examples

Question: A 9-year bond has a yield of 10% and a **Macaulay duration** of 7.194 years. If the market yield changes from 7% to 7.5% (a change of 50 basis points) what will be the percentage change in the bond's price?

ans: First we need to figure out by what percent the yield has changed. Moving from 7% to 7.5% means a $\frac{1.075-1.07}{1.07} = .0047 = 0.47\%$ change. Next we can find the percent change in the price $\frac{dP}{P} = -D * \frac{d(1+R)}{1+R} = -7.194 * 0.0047 = -0.0338 = -3.38\%$. A 3.38% drop in price.

Question: Same question but assume that the bond has a **modified duration** of 7.194 years now?

ans: Now we don't need to compute the percent change in the yield first, we just use the 50 basis point movement. $\frac{dP}{P} = -D * d(1+R) = -7.194 * 0.0050 = -0.0360 = -3.60\%$. A 3.60% drop in price.

2 Equity Valuation

2.1 Concepts

We can find the price of a stock similarly to the way we priced a bond. Just think of a stock as a stream of dividend payments occuring each period until you sell the stock at which point you receive a lump sum (the price of the stock at that point in time). One complication that arises however is that we don't know exactly how big the dividend payments or selling price will be, hence we need to think in terms of expected values.
The **dividend discount model** of stock valuation is,

$$\begin{aligned} P_t &= \frac{E_t[D_{t+1}]}{1+R} + \frac{E_t[D_{t+2}]}{(1+R)^2} + \cdots + \frac{E_t[D_{t+T}]}{(1+R)^T} + \frac{E_t[P_{t+T}]}{(1+R)^T} \\ &= \sum_{i=1}^{T} \frac{E_t[D_{t+i}]}{(1+R)^i} + \frac{E_t[P_{t+T}]}{(1+R)^T} \end{aligned}$$

assuming that R is a risk-adjusted discount rate appropriate for the stock that we are valuing. We will learn more about how to determine R later in the course.

Now if we assume that the price can't grow faster than the discount rate forever then,

$$\lim_{T\to\infty} \frac{E_t[P_{t+T}]}{(1+R)^T} = 0$$

and thus,

$$P_t = \sum_{i=1}^{\infty} \frac{E_t[D_{t+i}]}{(1+R)^i}$$

We can also consider the special case where we expect dividends to grow by the same amount each period,

$$\begin{aligned} E_t[D_{t+1}] &= D \\ E_t[D_{t+2}] &= (1+G)D \\ E_t[D_{t+3}] &= (1+G)^2 D \end{aligned}$$

$$\begin{aligned} P_t &= \sum_{i=1}^{\infty} \frac{E_t[D_{t+i}]}{(1+R)^i} \\ &= \sum_{i=1}^{\infty} \frac{(1+G)^{i-1} D}{(1+R)^i} \\ &= \frac{D}{1+R} \sum_{i=1}^{\infty} \frac{(1+G)^{i-1}}{(1+R)^{i-1}} \end{aligned}$$

Now assuming dividends grow slower than than the discount rate ($G < R$) then we can use our infinite sum formulas to get,

$$\begin{aligned} P_t &= \frac{D}{1+R} \cdot \frac{1}{1 - \frac{(1+G)}{(1+R)}} \\ &= \frac{D}{1+R} \cdot \frac{1+R}{R-G} \\ &= \frac{D}{R-G} \end{aligned}$$

which is the **Gordon growth model**.

2.2 Example

Question: Assume 10% is the appropriate discoutn rate to use for XYZ stock. What will the price of XYZ stock be if XYZ is expected to pay a dividend of \$5 next year, a dividend of \$5.25 the following year, and dividends are expected always grow at a constant rate?

ans: First calculate G, the constant dividend growth rate.

$$G = \frac{D_{t+2} - D_{t+1}}{D_{t+1}} = \frac{5.25 - 5}{5} = \frac{0.25}{5} = 5\%$$

Next, use the Gordon growth model to find the price of XYZ.

$$P_t = \frac{D}{R - G} = \frac{5}{10\% - 5\%} = 100$$