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April 18, 2007

Microeconomic Theory
Economics 101A
Spring 2007

Economics 101A: Microeconomic Theory
Section Notes for Week 13

1 Auction Example (Taken from Econ 201A ps 3 Fall 2004)

Consider a private value auction setup (where bidder i knows her own valuation, v_i, but not the other bidders' valuations). There are N bidders whose valuations are independent and uniformly distributed over the interval $[0, \alpha]$. Let $F_v(x) = Pr(v \leq x)$ be the cumulative distribution function and $f_v(x)$ the corresponding density funciton. Instead of any of the types of auctions we mentioned in class, consider an "all-pay" auction in which the highest bidder wins, but all bidders must pay their bid (even the losers).

1.1 Question 1

Find the equilibrium bidding function.

Assume that in equilibrium each bidder follows the same strategy and bids $b = B(v)$.

Let $v = g(b) = B^{-1}(b)$ denote the inverse bidding function.

For bidder i, the probability that another valuation, v, is lower than bidder i's is $F_v(v_i) = Pr(v \leq v_i)$.

The probability of winning the auction with bid b is the probability that all the other valuations (and thus all other bids) are lower than yours. To find this probability we note that the event that all other valuations are lower than yours is simply the intersection of the event that each particular other bid is lower than yours. Since the valuations are independent the probability of the union of these events can be expressed simply as the product of the probability of each particular event.

$$\Pi_{j=1,j\neq i}^{N} F_v(v_i) = [F_v(g(b_i))]^{N-1}$$

in this auction, bidder i must pay her bid b_i whether she wins or loses, thus the expected surplus to the bidder is

$$S(b; v) = Pr(Win) * valuation - bid = [F_v(g(b_i))]^{N-1} v_i - b_i$$

the bidder maximizes expected surplus which yields the FOC

$$\begin{aligned} \frac{\partial S(b;v)}{\partial b} &= \frac{\partial}{\partial b}[F_v(g(b_i))]^{N-1} v_i - b_i \\ &= v_i(N-1)[F_v(g(b_i))]^{N-2} \frac{\partial F_V(g(b_i))]}{\partial b} - 1 \\ &= v_i(N-1)[F_v(g(b_i))]^{N-2} f_v(v_i) g'(b_i) - 1 = 0 \end{aligned}$$

Note since the valuations have a uniform distribution $f_v(v_i) = 1/\alpha$, and $F_v(v_i) = v_i/\alpha$. Thus the FOC can be expressed as

$$[v_i(N-1)[v_i/\alpha]^{N-2}(1/\alpha)(1/B'(v_i)) - 1 = 0$$

remember that $(g'(b) = 1/B'(v))$ implying that

$$\begin{aligned}
B'(v_i) &= v_i(N-1)[v_i/\alpha]^{N-2}(1/\alpha) \\
&= (N-1)\left(\frac{v_i}{\alpha}\right)^{N-1} \\
B(v_i) &= (N-1)\int_0^{v_i}\left(\frac{v_i}{\alpha}\right)^{N-1} dv_i \\
&= \left[\frac{N-1}{N}\cdot\frac{v_i^N}{\alpha^{N-1}}\right]_0^{v_i} \\
&= \frac{N-1}{N}\cdot\frac{v_i^N}{\alpha^{N-1}}
\end{aligned}$$

1.2 Question 2

Compute the seller's expected revenue and compare it to the expected revenue from the English auction and second price sealed auctions discussed in class.

The expected revenue of this auction is

$$\begin{aligned}
E[Rev] &= NE[b_i] = NE[B(v_i)] \\
&= N\int_0^{\alpha} B(v_i)f_V(v_i)dv_i \\
&= N\int_0^{\alpha}\frac{N-1}{N}\cdot\frac{v_i^N}{\alpha^{N-1}}\cdot\frac{1}{\alpha}\cdot dv_i \\
&= (N-1)\int_0^{\alpha}\frac{v_i^N}{\alpha^N}\cdot dv_i \\
&= \frac{N-1}{N+1}\cdot\frac{\alpha^{N+1}}{\alpha^N} \\
&= \frac{N-1}{N+1}\cdot\alpha
\end{aligned}$$

Finding the expected revenue from the English and second price sealed bid auctions is a little bit trickier. In those auctions the winning bid will be equal to (or just above) the second highest valuation. We can denote this second highest valuation as $v_{(2)}$. Thus

$$E[Rev] = E[v_{(2)}]$$

but to evaluate this expectation we need to figure out the cumulative distribution function of $v_{(2)}$. Specifically we want to calculate

$$F_{V_{(2)}}(x) = Pr(\text{2nd highest bid} \leq x)$$

There are several ways that that the 2nd highest bid could be less than some value x. First, all bids could be less than x. Second, bidder i's bid could be greater than x and all other bids could be less than x. There are N possible ways the second case could happen since there are N bidders. Thus, the total probability is the union of all of these mutually exclusive events. Thus,

$$\begin{aligned} F_{V_{(2)}}(x) &= Pr(\text{all bids} \leq x) + \\ &\quad Pr(\text{1st bid} > x \text{ and all other bids} \geq x) + \\ &\quad Pr(\text{2nd bid} > x \text{ and all other bids} \geq x) + \cdots + \\ &\quad Pr(\text{Nth bid} > x \text{ and all other bids} \geq x) \\ &= [F_v(x)]^N + N(1 - F_v(x))[F_v(x)]^{N-1} \end{aligned}$$

differentiate to find the density

$$\begin{aligned} f_{V_{(2)}}(x) &= Nf_v(x)[F_v(x)]^{N-1} + N(N-1)f_v(x)[F_v(x)]^{N-2} - N^2 f_v(x)[F_v(x)]^{N-1} \\ &= N(N-1)f_v(x)[F_v(x)]^{N-2}(1 - F_v(x)) \end{aligned}$$

now we can evaluate the expectation

$$\begin{aligned} E[Rev] = E[v_{(2)}] &= \int_0^\alpha x f_{v_{(2)}}(x)dx \\ &= \int_0^\alpha xN(N-1)f_v(x)[F_v(x)]^{N-2}(1 - F_v(x))dx \\ &= N(N-1)\int_0^\alpha x \cdot \frac{1}{\alpha}\left(\frac{x}{\alpha}\right)^{N-2}\left(1 - \frac{x}{\alpha}\right)dx \\ &= N(N-1)\int_0^\alpha \left(\frac{x}{\alpha}\right)^{N-1} - \left(\frac{x}{\alpha}\right)^N dx \\ &= N(N-1)\left(\frac{\alpha^N}{N\alpha^{N-1}} - \frac{\alpha^{N+1}}{(N+1)\alpha^N}\right) \\ &= \alpha N(N-1)\left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= \alpha \cdot \frac{N-1}{N+1} \end{aligned}$$

whch is the same expected revenue as the all-pay auction.

2 Revenue Equivalence Theorem

In fact under the setup with N identical bidders with independent private valuations any auction with the following two properties with have the same expected revenue:

1. The bidder with the highest valuation wins.
2. The bidder with the minimum valuation possible in in the distribution has an expected payoff of zero.