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Microeconomic Theory
Economics 101A
Spring 2007

Economics 101A: Microeconomic Theory
Section Notes for Week 14: CAPM

1 Asset Demand in General

In order to think about the problem of the demand for financial assets in general we consider a consumer who derives utility through consumption and lives for two periods. A risky asset is available in the first period at a price p_t. The risky asset can then be sold in the second period at price p_{t+1} which is a random variable. The consumer begins the first period with W_t in initial wealth and must choose the quantity of the asset to purchase,s in order to maximize her total expected utility. This consumer's objective function can be written as

$$\max_s u(c_t) + \delta E_t[u(c_{t+1})]$$

where $0 < \delta < 1$ is the consumer's discount factor. The consumer faces two budget constraints. In the first period, she can either consume her wealth or invest in the asset and in the second period she can only consume as muct as the asset pays off.

$$\begin{aligned} c_t &= W_t - p_t s \\ c_{t+1} &= p_{t+1} s \end{aligned}$$

her first order condition with respect to the quantity of asset to buy is thus

$$\begin{aligned} \frac{\partial}{\partial s}\left(u(W_t - p_t s) + \delta E_t[u(p_{t+1}s)]\right) &= -p_t u'(W_t - p_t s) + \delta E_t[p_{t+1}u'(p_{t+1}s)] \\ &= -p_t u'(c_t) + \delta E_t[p_{t+1}u'(c_{t+1})] = 0 \end{aligned}$$

or

$$\begin{aligned} p_t u'(c_t) &= \delta E_t[p_{t+1}u'(c_{t+1})] \\ p_t &= E_t\left[\delta \frac{u'(c_{t+1})}{u'(c_t)} p_{t+1}\right] \end{aligned}$$

Note, in general, both $u'(c_{t+1})$ and p_{t+1} are random variables. The content of the CAPM or any asset pricing model is in specifying

$$m_{t+1} \equiv \delta \frac{u'(c_{t+1})}{u'(c_t)}$$

which is known as the stochastic discount factor (SDF).

We can also see what the first order condition implies about a risk-free asset.

$$\begin{aligned} p_t &= E_t\left[\delta \frac{u'(c_{t+1})}{u'(c_t)} p_t(1+R_f)\right] \\ &= p_t(1+R_f)E_t\left[\delta \frac{u'(c_{t+1})}{u'(c_t)}\right] \\ \frac{1}{1+R_f} &= E_t\left[\delta \frac{u'(c_{t+1})}{u'(c_t)}\right] = E_t[m_{t+1}] \end{aligned}$$

showing that the expected value of the SDF is equal to the inverse of the gross risk-free rate of return.

Also note that the pricing equation can be written as by divding both sides by p_t.

$$1 = E_t[m_{t+1}(1+R_{i,t+1})]$$

2 Content of The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) says two things

1. A linear factor pricing model holds.
2. The single factor in the model is the portfolio containing the entire asset market.

Quick Reminder: Assuming there are no dividend payments, the expected net return on asset i over the next year can is

$$E_t[R_{i,t+1}] = \frac{E_t[P_{i,t+1}] - P_{i,t}}{P_{i,t}}$$

re-arranging, we get

$$P_{i,t} = \frac{E_t[P_{i,t+1}]}{1 + E_t[R_{i,t+1}]}$$

thus an asset pricing model is equivalent to a model of expected returns.

3 Derivations of the CAPM

The following derivations are taken from John Cochrane's book, Asset Pricing (pp.152-167 1st edition). In lecture we saw a derivation of the CAPM in which investors had preferences for minimum variance portfolios given a target expected return. The following derivations assume that investors have more primative preferences defined over consumption. Note, these are all some sort of vN-M expected utilility preferences.

3.1 Two Period Quadratic Utility

In this derivation, investors have quadratic preferences over consumtion $u(c) = -\frac{1}{2}(c - c^*)^2$ and live for two periods.

$$\begin{aligned} U(c_t, c_{t+1}) &= u(c_t) + \delta E_t[u_(c_{t+1})] \\ &= -\frac{1}{2}(c_t - c^*)^2 - \delta \frac{1}{2} E_t[(c_{t+1} - c^*)^2] \end{aligned}$$

where $0 < \delta < 1$ is a discount factor. Note that c^* is what is referred to as the bliss point of the quadratic utility function. Remember, that it makes sense for consumers to prefer more wealth to less. The quadratic utility function looks like a parabola with the open part facing down, thus we have to be careful to only use the section of the parabola that is increasing. The bliss point, c^* represents the level of consumption at which the parabola acheives its maximum, thus when working with a quadratic utility function we want to think of c^* as some level of consumption far above anything that the consumer might actually realize.

In this model the SDF is

$$\begin{aligned} m_{t+1} &= \delta \frac{u'(c_{t+1})}{u'(c_t)} \\ &= \delta \left(\frac{c_{t+1} - c^*}{c_t - c^*} \right) \\ &= -\frac{\delta c^*}{c_t - c^*} + \left(\frac{\delta}{c_t - c^*} \right) c_{t+1} \\ &= a + b c_{t+1} \end{aligned}$$

which is a linear factor model where c_{t+1} is the factor. Thus we have shown part 1 of the CAPM. Next, we need to identify the factor. Since all remaining wealth is consumed in period two

$$c_{t+1} = (1 + R_{p,t+1})(W_t - c_t)$$

where the $R_{p,t+1}$ is the return on the optimal portfolio from the investor's FOC.

$$R_{p,t+1} = \sum_i \alpha_i^* R_{i,t+1} \text{ s.t. } \sum_i \alpha_i^* = 1$$

so the SDF can be written as

$$\begin{aligned} m_{t+1} &= \frac{\delta(W_t - c_t) - \delta c^*}{c_t - c^*} + \left(\frac{\delta(W_t - c_t)}{c_t - c^*} \right) R_{p,t+1} \\ &= a' + b' R_{p,t+1} \end{aligned}$$

Since all consumers have that same preferences, they will all hold the same portfolio, p. Since everyone wants to hold p, then in equilibrium all these shares of p must sum to be the entire asset market thus the portfolio p is a share of the entire market, so $R_{p,t+1} = R_{M,t+1}$ the market return. We have shown part two of the CAPM. The factor has been identified as the market portfolio.

We can pin down the parameters a' and b' by requiring that the model correctly price two assets such as the risk-free asset and the market protfolio.