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Microeconomic Theory
Economics 101A
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Economics 101A: Microeconomic Theory
Section Notes for Week 1

1 Multi-Variable Calculus

1.1 Partial Differentiation

The partial derivative of a multi-variable function $f(x_1, x_2)$ is the incremental change in the function caused by an incremental change in one of the variables while all other variables are held constant. The definition of the partial derivative of is

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h}$$

For convenience we sometimes write partial derivatives using the following alternative notations,

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = f_1(x_1, x_2) = f_{x_1}(x_1, x_2)$$

1.2 Total Differentiation

For small changes in the function $f(x_1, x_2)$ a first order Taylor approximation holds exactly, thus the total change in f is just the sum of the partials for each variable (slopes in each direction) times the incremental change in that particular variable.

$$\begin{aligned} df(x_1, x_2) &= \frac{\partial f(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial f(x_1, x_2)}{\partial x_2} dx_2 \\ &= f_1(x_1, x_2) dx_1 + f_2(x_1, x_2) dx_2 \end{aligned}$$

Now let's assume that x_2 is a known function of x_1, $x_2 = h(x_1)$. We could find the total derivative of $f(x_1, x_2)$ with respect to x_1 using the formula above

$$\begin{aligned} \frac{df(x_1, x_2)}{dx_1} &= \frac{df(x_1, h(x_1))}{dx_1} \\ &= f_1(x_1, h(x_1)) \frac{dx_1}{dx_1} + f_2(x_1, h(x_1)) \frac{dh(x_1)}{dx_1} \\ &= f_1(x_1, h(x_1)) + f_2(x_1, h(x_1)) \frac{dh(x_1)}{dx_1} \end{aligned}$$

1.3 Implicit Differentiation

Sometimes we may be in a situation where x_2 depends upon x_1 but there is no "explicit" formula that can be solved to express x_2 in terms of x_1, yet it is known that x_1 and x_2 satisfy some equation such as

$$f(x_1, x_2) = k$$

where k is a constant. Even though we can't explicitly solve for a function h such that $x_2 = h(x_1)$ it may still be possible to find the derivative, $\frac{dh(x_1)}{dx_1}$. To see this, start by taking the total derivative of the equation above with respect to x_1. The derivative of the left hand side (LHS) is

$$\frac{df(x_1, x_2)}{dx_1} = f_1(x_1, h(x_1)) + f_2(x_1, h(x_1))\frac{dh(x_1)}{dx_1}$$

The derivative of the right hand side (RHS) is

$$\frac{dk}{dx_1} = 0$$

thus we can rearrange and solve for $\frac{dh(x_1)}{dx_1}$

$$\begin{aligned} f_1(x_1, h(x_1)) + f_2(x_1, h(x_1))\frac{dh(x_1)}{dx_1} &= 0 \\ \frac{dh(x_1)}{dx_1} &= -\frac{f_1(x_1, h(x_1))}{f_2(x_1, h(x_1))} \end{aligned}$$

which is defined as long as $f_2(x_1, h(x_1)) \neq 0$.

2 Optimization

2.1 Unconstrained Optimization

Assuming that f has a maximum and it differentiable everywhere then a set of necessary conditions for a point in the interior of the function's domain, (x_1^*, x_2^*) to be a maximum are that the slope in each direction (the partial derivatives) of the function are zero at that point. Formally, the first order necessary conditions (FOC's) for (x_1^*, x_2^*) to solve,

$$\max_{x_1, x_2} f(x_1, x_2)$$

are

$$\begin{aligned} \frac{\partial f(x_1^*, x_2^*)}{dx_1} &= f_1(x_1^*, x_2^*) = 0 \\ \frac{\partial f(x_1^*, x_2^*)}{dx_2} &= f_2(x_1^*, x_2^*) = 0 \end{aligned}$$

Sufficient conditions to guarantee that (x_1^*, x_2^*) is a local maxima (as opposed to a local minima, or saddle point) involve the second partial derivatives of the function f at the point

(x_1^*, x_2^*). Specifically, if the FOC's hold and if

$$\begin{aligned} f_{11}(x_1^*, x_2^*) &< 0 \\ f_{22}(x_1^*, x_2^*) &< 0 \\ f_{11}(x_1^*, x_2^*) f_{22}(x_1^*, x_2^*) - f_{12}^2(x_1^*, x_2^*) &> 0 \end{aligned}$$

then (x_1^*, x_2^*) is a local maxima.

Sufficient conditions to guarantee that (x_1^*, x_2^*) is a global maxima of f (as opposed to a local maxima) involve the global shape of the function, or second partial derivatives of f at all points in f's domain. Specifically, if

$$\begin{aligned} f_{11}(x_1, x_2) &< 0 \\ f_{22}(x_1, x_2) &< 0 \\ f_{11}(x_1, x_2) f_{22}(x_1, x_2) - f_{12}^2(x_1, x_2) &> 0 \end{aligned}$$

for all possible points (x_1, x_2) in f's domain then (x_1^*, x_2^*) will be a global maxima.

2.2 Constrained Optimization

Suppose that we would like to find the maximum of $f(x_1, x_2)$ subject to the constraint $g(x_1, x_2) = k$.

2.2.1 Substition Solution

The first step in the substitution solution is to use the constraint equation to solve for x_2 as a function of x_1, or $x_2 = h(x_1)$. We can then rewrite the maximization problem as maximizing over only one choice variable, x_1.

$$\max_{x_1} f(x_1, h(x_1))$$

the first order necessary condition (FOC) comes from setting the total derivative equal to zero,

$$f_1(x_1^*, h(x_1^*)) + f_2(x_1^*, h(x_1^*)) \frac{dh(x_1^*)}{dx_1} \quad = \quad 0$$

We could then use this equation to solve for x_1^* and then $x_2^* = h(x_1^*)$.

Note, even if we could not solve for h explicity, the FOC implies that

$$\frac{dh(x_1^*)}{dx_1} = -\frac{f_1(x_1^*, h(x_1^*))}{f_2(x_1^*, h(x_1^*))}$$

and the constraint $(g(x_1, h(x_1)) = k)$ implies that

$$\frac{dh(x_1^*)}{dx_1} = -\frac{g_1(x_1^*, h(x_1^*))}{g_2(x_1^*, h(x_1^*))}$$

Putting the two together gives the familiar tangency condition

$$\frac{f_1(x_1^*, h(x_1^*))}{f_2(x_1^*, h(x_1^*))} = \frac{g_1(x_1^*, h(x_1^*))}{g_2(x_1^*, h(x_1^*))}$$

meaning that the constraint function, g, and the level curves of objective function, f, should be tangent at the optimium.

2.2.2 LaGrangian Solution

The LaGrangian method reformulates the constrained maximization as an unconstrained maximization problem with additional variables.

The first step is to form the LaGrangian

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda[g(x_1, x_2) - k]$$

the first order necessary condidtions (FOC's) are the same as they would be for the unconstrained maximization of $L(x_1, x_2, \lambda)$, or specifically

$$\begin{aligned}
L_1(x_1^*, x_2^*, \lambda) &= f_1(x_1^*, x_2^*) - \lambda g_1(x_1^*, x_2^*) = 0 \\
L_2(x_1^*, x_2^*, \lambda) &= f_2(x_1^*, x_2^*) - \lambda g_2(x_1^*, x_2^*) = 0 \\
L_3(x_1^*, x_2^*, \lambda) &= g_1(x_1^*, x_2^*) - k = 0
\end{aligned}$$

Note, the ratio of the first two equations imply the tangency condition

$$\frac{f_1(x_1^*, x_2^*)}{f_2(x_1^*, x_2^*)} = \frac{g_1(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}$$

Also note that the first two equations imply that

$$\lambda^* = \frac{f_1(x_1^*, x_2^*)}{g_1(x_1^*, x_2^*)} = \frac{f_2(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}$$

$lambda^*$ tells us the "shadow price" of the constriant $g(x_1, x_2) = k$ at the optimum, meaning the price we would pay to relax the constraint by raising k.

We can see this by thinking of the constraint level, k as a parameter, upon which the optimal levels of x_1, x_2, λ depend. Thus we can write the maximized objective function as a function of these optimal values

$$f(x_1^*(k), x_2^*(k), \lambda^*(k))$$

now we would like to see how this maximum changes as we relax the constraint by increasing k

$$\begin{aligned}
\frac{df(x_1^*(k), x_2^*(k))}{dk} &= f_1(x_1^*(k), x_2^*(k))\frac{dx_1^*(k)}{dk} + f_2(x_1^*(k), x_2^*(k))\frac{dx_2^*(k)}{dk} \\
&= \lambda^*(k) g_1(x_1^*(k), x_2^*(k))\frac{dx_1^*(k)}{dk} + \lambda^*(k) g_2(x_1^*(k), x_2^*(k))\frac{dx_2^*(k)}{dk} \\
&= \lambda^*(k)\left[g_1(x_1^*(k), x_2^*(k))\frac{dx_1^*(k)}{dk} + g_2(x_1^*(k), x_2^*(k))\frac{dx_2^*(k)}{dk}\right]
\end{aligned}$$

where the second equality is from obtained by plugging in the first two FOC's. Now take the total derivative of both sides of the constraint function with respect to k to obtain

$$\begin{aligned}\frac{dg(x_1^*(k), x_2^*(k))}{dk} &= \frac{d(k)}{dk} \\ g_1(x_1^*(k), x_2^*(k))\frac{dx_1^*(k)}{dk} + g_2(x_1^*(k), x_2^*(k))\frac{dx_2^*(k)}{dk} &= 1\end{aligned}$$

plugging this in, we get

$$\frac{df(x_1^*(k), x_2^*(k))}{dk} = \lambda^*(k)$$