Department of Economics
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Microeconomic Theory
Economics 101A
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Economics 101A: Microeconomic Theory
Section Notes for Week 3

1 The Envelope Theorem

Think back to the derivations of Roy's Identity and Shepard's Lemma (from week 2 section) and the interpretation of the LaGrange multiplier (from week 1 section).

LaGrange Multiplier as a shadow price of loosening the budget constraint.

$$\frac{\partial V}{\partial I} = \lambda^*(p_1, p_2, I)$$

Roy's Identity

$$\frac{\partial V}{\partial p_2} = -\lambda^*(p_1, p_2, I)x_1^*(p_1, p_2, I)$$

Shepard's Lemma

$$\frac{\partial e}{\partial p_2} = x_1^c(p_1, p_2, u^0)$$

The derivations of these identities are applications of the envelope theorem. To illustrate the envelope theorem, we return to the general constrained maximization setup from week 1 section, however instead of focussing solely on the endogenous variables (x_1, x_2), we also consider an exogenous variable ρ which can potentially affect the objective function f or the constraint function g. (In the context of the UMP or EMP the exogenous variable could be p_1, p_2, or I.)

$$\max_{x_1,x_2} f(x_1, x_2, \rho) \text{ s.t. } g(x_1, x_2, \rho) = k$$

The LaGrangian is then

$$L(x_1, x_2, \lambda, \rho) = f(x_1, x_2, \rho) - \lambda[g(x_1, x_2, \rho) - k]$$

and the first order necessary condidtions (FOC's)

$$\begin{aligned} L_1(x_1^*, x_2^*, \lambda, \rho) &= f_1(x_1^*, x_2^*, \rho) - \lambda^* g_1(x_1^*, x_2^*, \rho) = 0 \\ L_2(x_1^*, x_2^*, \lambda, \rho) &= f_2(x_1^*, x_2^*, \rho) - \lambda^* g_2(x_1^*, x_2^*, \rho) = 0 \\ L_3(x_1^*, x_2^*, \lambda, \rho) &= g_1(x_1^*, x_2^*, \rho) - k = 0 \end{aligned}$$

Note, there is no FOC from the partial derivative of the LaGrangian with respect to ρ since it is an exogenous variable, and thus can not be manipulated by the person doing the optimization. Solving the maximization leads to optimal values of the choice (or endogenous) variables as functions of the exogenous variable,

$$\begin{gathered} x_1^*(\rho) \\ x_2^*(\rho) \end{gathered}$$

if we were solving the UMP these would be the ordinary demand functions. Substituting these back into the objective function gives the **value function** (or indirect utility function for the UMP)

$$F(\rho) = f(x_1^*(\rho), x_2^*(\rho), \rho)$$

which is the maximum (or minimum if the objective is to minimze) possible value of the objective function given a certain value of the exogenous variable ρ. Now we can take the total derivative of the value function with respect to ρ to get

$$\frac{dF(\rho)}{d\rho} = \frac{\partial f}{\partial x_1^*} \cdot \frac{dx_1^*}{d\rho} + \frac{\partial f}{\partial x_2^*} \cdot \frac{dx_2^*}{d\rho} + \frac{\partial f}{\partial \rho}$$

Next, we can use the FOC's to substitute in for $\frac{\partial f}{\partial x_1^*}$ and $\frac{\partial f}{\partial x_2^*}$.

$$\begin{aligned} \frac{dF(\rho)}{d\rho} &= \lambda^* g_1(x_1^*, x_2^*, \rho) \cdot \frac{dx_1^*}{d\rho} + \lambda^* g_2(x_1^*, x_2^*, \rho) \cdot \frac{dx_2^*}{d\rho} + \frac{\partial f}{\partial \rho} \\ &= \lambda^* \left(g_1(x_1^*, x_2^*, \rho) \cdot \frac{dx_1^*}{d\rho} + g_2(x_1^*, x_2^*, \rho) \cdot \frac{dx_2^*}{d\rho} \right) + \frac{\partial f}{\partial \rho} \end{aligned}$$

we can also take the total derivative of the constraint $g_1(x_1^*, x_2^*, \rho) = k$ which gives

$$\begin{aligned} \frac{\partial g}{\partial x_1^*} \cdot \frac{dx_1^*}{d\rho} + \frac{\partial g}{\partial x_2^*} \cdot \frac{dx_2^*}{d\rho} + \frac{\partial g}{\partial \rho} &= 0 \\ \frac{\partial g}{\partial x_1^*} \cdot \frac{dx_1^*}{d\rho} + \frac{\partial g}{\partial x_2^*} \cdot \frac{dx_2^*}{d\rho} &= -\frac{\partial g}{\partial \rho} \end{aligned}$$

which can be substituted into the expression above to give

$$\frac{dF(\rho)}{d\rho} = -\lambda^*(\rho) \frac{\partial g(x_1^*, x_2^*, \rho)}{\partial \rho} + \frac{\partial f(x_1^*, x_2^*, \rho)}{\partial \rho}$$

remember this relationship holds at the optimum and that the optimal LaGrangian multiplier will generally be a function of the exogenous variable ρ. Note that this expression is equal to the partial derivative of the LaGrangian with respect to the exogenous variable evaluated at the optimum.

$$\frac{\partial L(x_1^*, x_2^*, \lambda^*, \rho)}{\partial \rho} = \frac{\partial f(x_1^*, x_2^*, \rho)}{\partial \rho} - \lambda^*(\rho) \frac{\partial g(x_1^*, x_2^*, \rho)}{\partial \rho} = \frac{dF(\rho)}{d\rho}$$

1.1 Example: Utility Maximization Problem (UMP)

In the UMP the objective is to maximize utility so

$$f(x_1, x_2, \rho) = u(x_1, x_2)$$

and the constriant is to do so within the budget so

$$g(x_1, x_2, \rho) = p_1x_1 + p_2x_2 - I$$

Next thing to note is that for the UMP we have a special name for the the value function. We call it the indirect utility function

$$F(\rho) = f(x_1^*(\rho), x_2^*(\rho), \rho) = V(p_1, p_2, I) = u(x_1^*(p_1, p_2, I), x_1^*(p_1, p_2, I))$$

Lets consider the impact of an exogenous change in p_1. We can evaluate the partials of f and g with respect to the exogenous variable p_1 at the optimum.

$$\begin{aligned}\frac{\partial f(x_1^*, x_2^*, \rho)}{\partial \rho} &= \frac{\partial u(x_1^*, x_2^*)}{\partial p_1} = 0 \\ \frac{\partial g(x_1^*, x_2^*, \rho)}{\partial \rho} &= \frac{\partial (p_1x_1^* + p_2x_2^*)}{\partial p_1} = x_1^*\end{aligned}$$

then using the envelope condition we get

$$\frac{dV}{dp_1} = 0 - \lambda^* x_1^*$$

which is Roy's Identity and which we dervied in week 2 of section.

1.2 Example: Expense Minimization Problem (UMP)

In the EMP the objective is to minimize expenditures so

$$f(x_1, x_2, p_1) = p_1x_1 + p_2x_2$$

and the constriant is to do so with in the budget so

$$g(x_1, x_2, p_1) = u(x_1, x_2) = u^0$$

For the EMP we also have a special name for the value function. We call it the expenditure function.

$$F(\rho) = f(x_1^*(\rho), x_2^*(\rho), \rho) = e(p_1, p_2, u^0) = p_1x_1^c(p_1, p_2, u^0) + p_2x_2^c(p_1, p_2, u^0)$$

Lets consider the impact of an exogenous change in p_1. We can evaluate the partials of f and g with respect to the exogenous variable p_1 at the optimum.

$$\begin{aligned}\frac{\partial f(x_1^*, x_2^*, \rho)}{\partial \rho} &= \frac{\partial (p_1x_1^c + p_2x_2^c)}{\partial p_1} = x_1^c \\ \frac{\partial g(x_1^*, x_2^*, \rho)}{\partial \rho} &= \frac{\partial u(x_1^c, x_2^c)}{\partial p_1} = 0\end{aligned}$$

then using the envelope condition we get

$$\frac{de}{dp_1} = x_1^c - \lambda^* \cdot 0 = x_1^c$$

which is Shepard's Lemma.

2 Other Identities

$$e(p_1, p_2, V(p_1, p_2, I)) = I$$

Meaning that the minimal expenditure needed to achieve the maximal amount of utility that can be achieved at prices p_1, p_2 and income I is I.

$$V(p_1, p_2, e(p_1, p_2, u^0)) = u^0$$

Meaning that that maximal amount of utility that can be achieved at prices p_1, p_2 and income level equal to the minimal level of expenditure required to achieve utility u^0 is (of course) u^0.

$$\begin{aligned} x_1^c(p_1, p_2, u^0) &= x_1(p_1, p_2, e(p_1, p_2, u^0)) \\ x_2^c(p_1, p_2, u^0) &= x_2(p_1, p_2, e(p_1, p_2, u^0)) \end{aligned}$$

Meaning that ordinary and compensated demand will be equal if the amount of income that the consumer has happens to be equal to the minimal amount necessary to achieve utility level u^0. (Note I am denoting the ordinary demand function with no superscript.)

$$\begin{aligned} x_1(p_1, p_2, I) &= x_1^c(p_1, p_2, V(p_1, p_2, I)) \\ x_2(p_1, p_2, I) &= x_2^c(p_1, p_2, V(p_1, p_2, I)) \end{aligned}$$

Meaning that ordinary and compensated demand will be equal if the utility level that the consumer must achieve is equal to the maximal utility level possible given that they have income I.

2.1 Example

Let's examine the UMP for the specific utility function $u(x_1, x_2) = x_1 + \log x_2$

$$\max_{x_1, x_2} x_1 + \log x_2 \text{ s.t. } p_1 x_1 + p_2 x_2$$

The FOCs are

$$\begin{aligned} 1 - \lambda^* p_1 &= 0 \\ \frac{1}{x_2^*} - \lambda^* p_2 &= 0 \\ p_1 x_1^* + p_2 x_2^* &= I \end{aligned}$$

The tangency condition is

$$\frac{p_1}{p_2} = x_2^*$$

The ordinary demand functions are

$$\begin{aligned} x_1(p_1, p_2, I) &= \frac{I - p_1}{p_1} \\ x_2(p_1, p_2, I) &= \frac{p_1}{p_2} \end{aligned}$$

Pluggin the ordinary demand functions into the utility function we get the indirect utility function

$$V(p_1, p_2, I) = \frac{I - p_1}{p_1} + \log\left(\frac{p_1}{p_2}\right)$$

Now as long as it is easy to solve for I we can use our identities from above to find the expenditure function quickly

$$\begin{aligned}
V(p_1, p_2, e(p_1, p_2, u^0)) &= u^0 \\
\frac{e(p_1, p_2, u^0) - p_1}{p_1} + \log\left(\frac{p_1}{p_2}\right) &= u^0 \\
e(p_1, p_2, u^0) &= p_1\left(u^0 - \log\left(\frac{p_1}{p_2}\right)\right) + p_1
\end{aligned}$$

Furthermore, if we didn't feel like using Shepard's Lemma and taking the derivative of the expenditure function to find the compensated demand functions, we could use the above identities for compensated demands

$$\begin{aligned}
x_1^c(p_1, p_2, u^0) &= x_1(p_1, p_2, e(p_1, p_2, u^0)) = \frac{e(p_1, p_2, u^0) - p_1}{p_1} = u^0 - \log\left(\frac{p_1}{p_2}\right) \\
x_2^c(p_1, p_2, u^0) &= x_2(p_1, p_2, e(p_1, p_2, u^0)) = \frac{p_1}{p_2}
\end{aligned}$$

3 Slutsky Equation

3.1 Easy Derivation

The Slutsky equation provides a decomposition of how an optimizing consumer reacts to an small incremental change in prices. To derive the Slutsky equation, begin with the identity

$$x_1^c(p_1, p_2, u^0) = x_1(p_1, p_2, e(p_1, p_2, u^0))$$

Take the partial derivative with respect to p_1 of both sides

$$\frac{\partial x_1^c(p_1, p_2, u^0)}{\partial p_1} = \frac{\partial x_1(p_1, p_2, e(p_1, p_2, u^0))}{\partial p_1} + \frac{\partial x_1(p_1, p_2, e(p_1, p_2, u^0))}{\partial e} \cdot \frac{\partial e(p_1, p_2, u^0)}{\partial p_1}$$

but since $e(p_1, p_2, u^0) = I$ at the optimum and from Shepard's Lemma $\frac{\partial e(p_1, p_2, u^0)}{\partial p_1} = x_1^c$

$$\frac{\partial x_1^c(p_1, p_2, u^0)}{\partial p_1} = \frac{\partial x_1(p_1, p_2, I)}{\partial p_1} + \frac{\partial x_1(p_1, p_2, I)}{\partial I} \cdot x_1^c(p_1, p_2, u^0)$$

Finally, noting that $x_1(p_1, p_2, I) = x_1^c(p_1, p_2, u^0)$ at the optimum and re-arranging gives

$$\frac{\partial x_1(p_1, p_2, I)}{\partial p_1} = \frac{\partial x_1^c(p_1, p_2, u^0)}{\partial p_1} - \frac{\partial x_1(p_1, p_2, I)}{\partial I} \cdot x_1(p_1, p_2, I)$$

The interpretation of the Slutsky equation is that we can decompose an the response consumer response to a change in prices into substitution and income effects. Where

$$\frac{\partial x_1^c(p_1, p_2, u^0)}{\partial p_1}$$

is the substitution effect which can be interpretted as as small movement along the indifference curve u^0, and

$$-\frac{\partial x_1(p_1,p_2,I)}{\partial I} \cdot x_1(p_1,p_2,I)$$

is the income effect. Note that the partial $\frac{\partial x_1(p_1,p_2,I)}{\partial I} > 0$ for normal goods. And that the Slutsky equation applies to interior optima so that $x_1(p_1,p_2,I) > 0$.

As Professor Card mentioned in class, the Slutsky equation describes how a consumer who is already optimizing at an interior point (consuming a positive amount of both goods) would change their demand in response to a change in price, and does not apply to consumer entry into the market.