# Dominance and Decisions

Kenny Easwaran

Group in Logic and the Methodology of Science
University of California, Berkeley

July 4, 2006

# Introduction



- Classical decision theory decides between actions by calculating their expected utilities.
- But various paradoxes like St. Petersburg, Pasadena, Ellsberg, Two-Envelope, and the like have shown some of the limits of expected utility reasoning.
- I would like to develop a new decision theory based instead on dominance reasoning - this is the beginnings of that project.
- Much of this theory is an elaboration of ideas in [Dietrich and List, "The Two-Envelope Paradox: An Axiomatic Approach"], though they didn't put as much emphasis on dominance.

# St. Petersburg and Pasadena



- I will repeatedly flip a (fair) coin until it comes up heads - the payoff will depend only on $n$, the number of flips needed.
- In St. Petersburg, the payoff is $\$2^n$, while in Pasadena it is $\$\frac{(-2)^n}{n}$. [Nover and Hájek, "Vexing Expectations"]
- The expected utility of St. Petersburg is thus infinite, while that of Pasadena is undefined. (In one order, the sum comes to $\log 2$, but it can sum to any value in $[-\infty, +\infty]$.)

# Leningrad and Altadena



- Despite these problematic expectations, it seems that we can make some decisions.
- The Leningrad game [Colyvan, "Relative Expectation Theory"] has payoffs $\$(2^n + 1)$ while the Altadena game has payoffs $\$\left(\frac{(-2)^n}{n} + 1\right)$.
- Although the expectations are respectively still $\infty$ and undefined, it seems clear that one should prefer Leningrad to St. Petersburg and Altadena to Pasadena, because the same state always leads to a greater payoff in the former cases.
- This is dominance reasoning. In most cases it is much weaker than expected utility, but here it speaks where expected utility can't.

# Dominance Failure



- However, now consider St. Petersburg played by flipping until heads comes up, and Leningrad played by flipping until tails comes up. (Or a similar Pasadena/Altadena pair.)
- Now, the notion of "the same state" doesn't seem to apply at all and we can't apply dominance reasoning.
- Expected utility can't tell us anything except indifference.
- But we should clearly prefer Leningrad and Altadena.
- [Colyvan, "Relative Expectation Theory"] has suggested generalizing expected utility. My full-fledged theory will in fact extend expected utility theory, but it starts with dominance as the basis instead.

# Background



- The standard framework of decision theory involves a set $\mathcal{A}$ of *actions*, a set $\mathcal{S}$ of *states*, and a set $\mathcal{O}$ of *outcomes*.
- Each state is also (generally) associated with a probability (a real number in $[0, 1]$), and each outcome with a real-number utility.
- Given an action and state, there is a unique outcome. In particular, an action $a \in \mathcal{A}$ is considered to be a function $a \colon \mathcal{S} \to \mathcal{O}$.
- Importantly, the states are assumed to be independent of the actions.

# Different State Spaces

- The examples of heads versus tails St. Petersburg suggests that we sometimes don't have the same state space for all actions - had I made a different choice, the coin might not have been flipped at all!
- Without the same state space, we can calculate expected utilities (assuming probabilities in each space are well-defined, and we don't run into St. Petersburg or Pasadena type problems), but we can't use dominance reasoning directly.
- Finding the right way to identify states, so I can extend dominance reasoning, may shed some light on other decision problems as well.

# Prisoner's Dilemma



- There are two players - each can either choose to *cooperate* or *defect*.
- If both cooperate, they each receive 3 units of utility, while if both defect they each receive 1 unit.
- If one cooperates and the other defects, then the one that defects gets 4 units, while the one that cooperates gets 0.
- Thus, it seems that whichever action one player chooses, the other gets one more unit for defecting.
- Dominance thus seems to tell both to defect, even though they'd be better off if both cooperated.

# Frederic Schick, *Ambiguity and Logic*, "A Dilemma for Whom?"



- Schick points out that this dominance requirement only arises for either player if she sets up her decision problem with the state space being the actions $C$ and $D$ of the other player.
- Instead she can consider the state space to be $\mathcal{S} = \{A, O\}$, "*Jill will do as I will* ($A$) or *She will do the opposite* ($O$)". (p. 22)
- Now, in state $A$, it is better to cooperate, and in state $O$ it is better to defect, so there is no dominance.
- So which state space is correct? $\{C, D\}$ or $\{A, O\}$?

# Frederic Schick, *Ambiguity and Logic*, "A Dilemma for Whom?"



- In $\{C, D\}$, dominance reasoning requires one to defect, but in $\{A, O\}$, pure dominance reasoning remains silent.
- We normally take the states to represent something about the world that is in some sense independent of the agent's action.
- $\{C, D\}$ seem to be most directly causally independent of the agent's action.
- But $\{A, O\}$ may well be probabilistically independent - if our agent is "ideally rational", then she may let $A$ represent the possibility that the opponent is as well, and $O$ be otherwise.

# Newcomb's Paradox



- A wealthy fortune teller has announced that she is giving away some of her money today, in boxes A and B.
- You may either just take box A, or take both.
- She has put $1000 in box B, but the amount in box A depends on whether she has predicted you will take just A or will take both.
- If she thinks you will just take A, then she has put $1,000,000 in A, but if she thinks you are greedy and will take both, then she has put nothing in A.
- You have observed her in this set-up many times before, and she is 90% accurate in her predictions, both of people one-boxing and two-boxing.

# Evidential and Causal Decision Theory



- Evidential decision theorists: if you one-box, there is a 90% chance of \$1,000,000 and a 10% chance of \$0. Expectation is \$900,000.
- If you two-box, there is a 90% chance of \$1000 and a 10% chance of \$1,001,000. Expectation is \$101,000.
- Therefore, expected utility says you should one-box.

# Evidential and Causal Decision Theory



- Evidential decision theorists: if you one-box, there is a 90% chance of $1,000,000 and a 10% chance of $0. Expectation is $900,000.
- If you two-box, there is a 90% chance of $1000 and a 10% chance of $1,001,000. Expectation is $101,000.
- Therefore, expected utility says you should one-box.
- Causal decision theorists: the states are *Money* and *No Money*, and in either state, two-boxing gives you $1000 more, so dominance says you should two-box.

# Different State Spaces



- Many have taken this Newcomb paradox to show that expected utility is in conflict with dominance reasoning.
- However, it may be reasonable to take the states to be *Correct Prediction* and *Incorrect Prediction*, in which case dominance reasoning says nothing.
- Which is the correct state space will depend on what sort of independence is required for states - causal independence might suggest $\{M, N\}$ while probabilistic independence might suggest $\{C, I\}$.
- I will (very tentatively) suggest probabilistic independence as the correct choice, because it carries less metaphysical baggage - fortunately, the two tend to agree in cases other than these.

# St. Petersburg and Leningrad

- We can use a similar strategy even when there *is* a unique well-defined state space. Say a coin will be flipped until it has come up both heads and tails at least once each.
- Action $L_H$ has payoff $\$(2^m + 1)$ if the first *head* occurs at flip $m$.
- Action $S_T$ has payoff $\$2^n$ if the first *tail* occurs at flip $n$.

| State | ... | HHHT | HHT | HT | TH | TTH | TTTH | ... |
|---|---|---|---|---|---|---|---|---|
| $L_H$ | ... | \$3 | \$3 | \$3 | \$5 | \$9 | \$17 | ... |
| $S_T$ | ... | \$16 | \$8 | \$4 | \$2 | \$2 | \$2 | ... |

# St. Petersburg and Leningrad

- We can use a similar strategy even when there *is* a unique well-defined state space. Say a coin will be flipped until it has come up both heads and tails at least once each.
- Action $L_H$ has payoff $\$(2^m + 1)$ if the first *head* occurs at flip $m$.
- Action $S_T$ has payoff $\$2^n$ if the first *tail* occurs at flip $n$.

| State | ... | HHHT | HHT | HT | TH | TTH | TTTH | ... |
|---|---|---|---|---|---|---|---|---|
| $L_H$ | ... | \$3 | \$3 | \$3 | \$5 | \$9 | \$17 | ... |
| $S_T$ | ... | \$16 | \$8 | \$4 | \$2 | \$2 | \$2 | ... |

- If the coin is fair, then we can interchange all heads and tails to give an automorphism of the state space that preserves probabilities, and see that $L_H$ dominates $S_T$ on the new identification.

# The Ellsberg Paradox



- There is an urn with 90 balls that are all red, black, or white, and exactly 30 of them are red.
- Would you prefer $(A)$ \$100 if a randomly drawn ball is red, or $(B)$ \$100 if the randomly drawn ball is black?
- Would you prefer $(C)$ \$100 if the randomly drawn ball is red or white, or $(D)$ \$100 if the randomly drawn ball is black or white?

# The Ellsberg Paradox



- There is an urn with 90 balls that are all red, black, or white, and exactly 30 of them are red.
- Would you prefer ($A$) \$100 if a randomly drawn ball is red, or ($B$) \$100 if the randomly drawn ball is black?
- Would you prefer ($C$) \$100 if the randomly drawn ball is red or white, or ($D$) \$100 if the randomly drawn ball is black or white?
- Most people (strictly) prefer $A$ and $D$, even though this contradicts any assignment of probabilities to the randomly drawn ball being black or white (as well as Savage's "sure-thing" principle).

# The Dominance-Based Approach



- Thus, the Ellsberg paradox is a problem for expected utility theory, if people are assumed to have some implicit probability assignments.
- It seems that they instead assign no particular probabilities to black and white.
- On my dominance account, no permutation of the state space will be allowed to identify events of the ball being red, black, or white with one another, or with any of their sub-events, because they don't have the same (or indeed *any*) probabilities.
- Some non-dominance rule will be needed as well here to positively achieve the intuitive choices, but at least they aren't contradicted.

# "Spreading Poison"

- Alan Hájek has suggested a further problem involving the Pasadena game - when making an ordinary decision, say about whether to have Indian or Chinese food for dinner, we may not be able to rule out the Pasadena game happening.
- If the expectation of the Pasadena game is really undefined, and expected utility theory can't tell us what to do if the utilities are undefined, then expected utility theory can never help us if the Pasadena game has non-zero probability.
- To make matters worse, some authors have suggested that we should *always* assign non-zero probability to any non-contradictory statement.

# The Dominance-Based Solution



- However, this problem is solved by a dominance-based account supplemented by these permutations.
- In ordinary cases, it seems that the possibility of running into the Pasadena game can reasonably be considered independent of whether you choose Indian or Chinese for dinner, even though the other relevant states might be different.
- Thus, when identifying states between the two actions, we can line up the states where one runs into Pasadena so they "cancel out", and end up making the decision based only on the issues that seem intuitively relevant.
- But we may still need something like expected utility machinery to make the rest of this decision.

# Shifting utility

- Consider the choice between guaranteed \$0, and a flip of a fair coin between \$1,000,000 and a loss of \$1.
- Clearly, one should choose the coin flip, but no rearrangement of states allows dominance here.
- Since we are allowing permutations of states that have the same probability, the natural next step is to allow the movement of some amount of utility from one state to another state of the same probability.
- In this case, we can ensure dominance by moving \$2 from the million dollar state to the state with the loss.
- With this rule, we can basically get any expected utility judgement for gambles with only finitely many states.

# Relative expectation





- Another possible approach (suggested by David Chalmers in questions at a previous presentation) is to follow [Colyvan, "Relative Expectation Theory"], let $D_s$ be the difference in payoffs in (properly identified) state $s$, and then take the expectation of $D_s$.
- The re-identification of states is necessary to deal with a decision between heads St. Petersburg and tails St. Petersburg, whose relative expectation with the standard identification of states is given by the sum $1 - 1 + 1 - 1 + 1 - \ldots$.

# Deciding between the extensions



- Relative expectation does better than utility shifting by allowing a decision in cases involving certain "well-behaved" infinities, like a choice between nothing and a gamble that pays $\$((-1)^n n)$ if the fair coin takes $n$ flips to reach heads.
- Since the expectation of the latter is $\$-2/9$, it's best not to take the gamble.

# Deciding between the extensions





- Relative expectation does better than utility shifting by allowing a decision in cases involving certain "well-behaved" infinities, like a choice between nothing and a gamble that pays $$((-1)^n n)$ if the fair coin takes $n$ flips to reach heads.
- Since the expectation of the latter is $\$-2/9$, it's best not to take the gamble.
- However, relative expectation wipes out any differences that occur on a set of probability 0, which one may not want to do.
- So it is unclear which is better.

# States vs. Events



- An important distinction in probability theory, and therefore in decision theory, is that between *states* and *events*.
- A state is something like a maximal consistent description of the world that is independent of the actions of the agent, while an event is less complete.
- Formally, we can say that an event is any subset of the state space. (Though we may sometimes limit ourselves to a particular collection, like the "measurable" subsets.)
- What I have called "states" in earlier discussions are perhaps better called "events", but since the outcome was uniform over all states in each event, we still had statewise dominance.

# Savage's "Sure-Thing" Principle



- We know what it is to prefer one outcome to another, and therefore can say when we prefer one action to another, *given a state*.
- Decision theory aims to tell us when to prefer one action to another *simpliciter*.
- We can also introduce a theoretical notion of preferring one action to another *given an event*.
- One way to put Savage's "sure-thing" principle is as statewise dominance.
- Given a mutually disjoint and exhaustive set $E$ of events, an agent who prefers action $A$ to action $B$ *given each event* in $E$ should prefer $A$ to $B$ *simpliciter*.

# Statewise and Eventwise Dominance



- Statewise and eventwise dominance are often confused for one another.
- However I, like Dietrich and List (in "The Two-Envelope Paradox: An Axiomatic Approach"), suggest that dominance reasoning is a good decision procedure, while the sure-thing principle is inconsistent.
- The relevant case is an instance of the two-envelope paradox, and the position I endorse is that of [Chalmers, "The St. Petersburg Two-Envelope Paradox"], though he suggests that it counts *against* dominance. (Chalmers' resolution is anticipated by [Arntzenius and McCarthy, "The Two Envelope Paradox and Infinite Expectations"], though Chalmers focuses more specifically on dominance principles.)
- I suggest it only counts against *eventwise* dominance, not *statewise* dominance.

# The Two-Envelope Paradox



- The agent is presented with two envelopes, and is told that one has twice as much money as the other.
- The agent chooses an envelope, sees that it contains $\$x$, and infers that the other either contains $\$2x$ or $\$x/2$, and so has an expectation of $\$5x/4$.
- Given a choice between keeping the first envelope, or switching, the agent would prefer to switch.
- Since this is independent of the amount in the envelope, it seems that one should switch even before seeing the amount, which is clearly a paradox.

# The Two-Envelope Paradox with a Prior



- [Jackson, Menzies, and Oppy, "The Two Envelope 'Paradox'"] have suggested that if one has a prior distribution for the amount of money in either envelope, then the problem will go away.
- Presumably, updating by conditionalizing on the amount seen in the envelope will in some cases give the other envelope higher expectation, and sometimes lower.
- However, [Broome, "The Two-Envelope Paradox"] shows that if we assign probability $\frac{(2/3)^n}{3}$ to the envelopes having $\$2^n$ and $\$2^{n+1}$, then the expectation of switching is still always higher.

# Eventwise Dominance Contradicts Expected Utility

- This set-up leads immediately to a proof that eventwise dominance is false for such cases, given some conditional expected utility reasoning.
- Partitioning the state space into events $E_i$ where the *left* envelope has \$$2^i$, finitary expected utility theory (which I will grant for now) shows that one prefers the right envelope given each $E_i$, so one should prefer the right envelope unconditionally.
- Partitioning into $F_i$ where the *right* envelope has \$$2^i$, one prefers the left envelope given each $F_i$, so one should prefer the left envelope unconditionally.
- This is a contradiction.

# The Two-Envelope Paradox Resolved



- This is Chalmers' solution - although one should prefer to switch once one looks inside the envelope, one should be indifferent before then.
- This contradicts the sure-thing principle, but not statewise dominance.
- He illustrates the non-paradoxicality of this outcome by considering two envelopes independently filled by St. Petersburg processes.
- Unconditionally, it seems one should be indifferent between the two, but once one knows the amount in L, one should automatically prefer the infinite expectation of R.

# The Dominance-Based Approach

- Dominance-based reasoning can reach the same conclusions.
- With *or without* a prior we can permute the states "L has $\$2^n$ and R has $\$2^{n+1}$" and "R has $\$2^n$ and L has $\$2^{n+1}$" to show that one should be indifferent between R and L unconditionally. (This is the main innovation of Dietrich and List.)
- With a prior, once one sees the amount in L is $\$2^n$, there will be probability $\frac{p_{n-1}}{p_{n-1}+p_n}$ of R having $\$2^{n-1}$ and $\frac{p_n}{p_{n-1}+p_n}$ of R having $\$2^{n+1}$.
- If I allow movement of finite amounts of utility between states of the same probability, just as I allow permutation of states of the same probability, then I can agree with any expected utility judgement based on finitely many outcomes with well-defined probabilities.

# Conclusion

- Thus, versions of dominance-based reasoning can deal with any ordinary case that expected utility reasoning can, though the best way to do so remains unclear.
- But dominance-based reasoning can also deal with problematic cases, giving the correct preferences between infinitary cases like Altadena and Pasadena, even in different state spaces.
- It can also give the correct judgements in probability-free cases, like the Two Envelope paradox without a prior, and the Ellsberg paradox.
- There are some remaining problems with identifying states across spaces (I have tentatively assumed this is done by matching probabilities), but proper solutions to this problem will also shed light on the Prisoner's Dilemma and Newcomb Paradox.