Problem Solving Through Problems: Loren C. Larson

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1. Consider the following argument. Suppose θ satisfies #1.8.8 on page 44

$$\cot\theta + \tan(3\theta) = 0.$$

Then, since

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

it follows that

$$\begin{aligned}
\cot\theta + \frac{\tan\theta + \tan(2\theta)}{1 - \tan\theta\tan(2\theta)} &= 0,\\
\cot\theta\,(1 - \tan\theta\tan(2\theta)) + \tan\theta + \tan(2\theta) &= 0,\\
\cot\theta - \tan(2\theta) + \tan\theta + \tan(2\theta) &= 0,\\
\cot\theta + \tan\theta &= 0,\\
1 + \tan^2\theta &= 0,\\
\tan^2\theta &= -1.
\end{aligned}$$

Since this last equation cannot hold, the original equation does not have a solution. What is wrong with this argument?

2. Show that $x^7 - 2x^5 + 10x^2 - 1$ has no root greater than 1. (Hint: Since it is generally easier to show that an equation has no positive root, we are prompted to consider the equivalent problem obtained by making the algebraic substitution $x = y + 1$.) #1.3.8 on page 20

3. Sum the infinite series #5.4.4 on page 180

$$\frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots.$$

Solution 1 *Begin with the series*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Multiply each side by x to get:

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}.$$

Differentiate each side to get:

$$(1+x)\,e^x = \sum_{n=0}^{\infty} \frac{(n+1)\,x^n}{n!}.$$

Now multiply each side by x again:

$$\left(x+x^2\right)e^x = \sum_{n=0}^{\infty} \frac{(n+1)\,x^{n+1}}{n!}.$$

Differentiate each side one more time:

$$\left(1+3x+x^2\right)e^x = \sum_{n=0}^{\infty} \frac{(n+1)^2\,x^n}{n!}.$$

Setting $x=1$, we see that

$$\sum_{n=0}^{\infty} \frac{(n+1)^2\,x^n}{n!} = \frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots = 5e.$$

4. Show that #3.5.8 on page 118

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \cdots = 2^{n/2}\cos\left(\frac{n\pi}{4}\right)$$

and

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{n/2}\sin\left(\frac{n\pi}{4}\right).$$

[Hint: Consider $(1+i)^n$.]

5. Determine all solutions in real numbers x, y, z, w of the system #4.1.2 on page 121

$$\begin{aligned} x+y+z &= w, \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z} &= \frac{1}{w}. \end{aligned}$$

Solution 2 *Simplifying the second equation, we get*

$$\frac{yz+xz+xy}{xyz} = \frac{1}{w}.$$

Multipliying this by the first equation gives

$$(x+y+z)\,(yz+xz+xy) = xyz.$$

This expands to

$$x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + 2xyz = 0,$$

which factors into

$$(x+y)\,(x+z)\,(y+z) = 0.$$

Thus $x=-y$ or $x=-z$ or $y=-z$. If $y=-z$, we get $x=w$. The argument is similar for the other cases.