Math 160 Paper 1: Diophantus and Number Theory

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1 Introduction

The known personal information of Diophantus is best summed up in the following quote: "Here you see the tomb containing the remains of Diophantus, it is remarkable: artfully it tells the measures of his life. The sixth part of his life God granted him for his youth. After a twelfth more his cheeks were bearded. After an additional seventh he kindled the light of marriage, and in the fifth year he accepted a son. Elas, a dear but unfortunate child, half of his father he was and this was also the span a cruel fate granted it, and he consoled his grief in the remaining four years of his life. By this device of numbers, tell us the extent of his life." With our modern notation and use of algebra, it is rather easy to see that if Diophantus lived for x years, then the problem is equivalent to solving the equation $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x$, which has the solution $x = 84$.

It is hard to pinpoint when exactly Diophantus lived. The Arabian historian Abū'lfaraj mentions in his *History of the Dynasties* that Diophantus lived during the time of Emperor Julian ($361 - 363$ A.D.), whereas Rafael Bombelli says authoritatively in his book, *Algebra*, that Diophantus lived during the reign of Antoninus Pius ($138 - 161$ A.D.). There is little or no confirmation of either of these dates. However, in his *On Polygonal Numbers*, Diophantus defines a polygonal number by quoting Hypsicles. Hypsicles was the writer of the supplement to Book XIII of Euclid's *Elements*, and so Diophantus must have written after about 150 B.C. Furthermore, he is quoted by Theon of Alexandria, so we infer that he wrote before 350 A.D. There is also a letter by Psellus, who lived in the 11^{th} century, wherein it is stated that Diophantus and Anatolius were writers on the Egyptian method of reckoning. Psellus says, "Diophantus dealt with it more accurately, but the very learned Anatolius collected the most essential parts of the doctrine as stated by Diophantus in a different way and in the most succinct form, dedicating his work to Diophantus." From here it appears that Diophantus and Anatolius lived around the same time, and their relationship was akin to that of a master and his disciple. Anatolius was the Bishop of Laodicea around 280 A.D., so it seems appropriate to conclude that Diophantus flourished around 250 A.D.

Diophantus dedicated *Arithmetica* to Dionysius: "Knowing, my most esteemed friend Dionysius, that you are anxious to learn how to investigate problems in numbers, I have tried, beginning from the foundations on which the science is built up, to set forth to you the nature and power subsisting in numbers." A certain Dionysius was the director of Alexandria's Christian high school from 231 A.D., and eventually became bishop of Alexandria in 247 A.D. Paul Tannery, the editor of the most complete text of Diophantus, identifies each Dionysius as the same person, and similarly deduces that Diophantus lived around the middle of the third century. The one thing we know for sure about Diophantus is that he lived in Alexandria, the hub of scientific activity in the Hellenic world.

2 Mathematical Work

Of all the work Diophantus possibly did, only four writings have survived: The *Arithmetica*, a tract *On Polygonal Numbers*, *Porismata*, and *Moriastica*.

The *Moriastica* is mentioned only in a scholium to Iamblichus' commentary on Nicomachus' *Arithmetica*. It seems as if it was a separate work by Diophantus with rules for computing with fractions.

The *Porismata* is a collection of propositions dealing with the properties of certain numbers, their divisibility into a certain number of squares, and so on. Thomas Heath believes that this text is an independent work by Diophantus, whereas Tannery adopts the view that the *Porismata* is a collection of lemmas included with the *Arithmetica*.

Tannery believes that just as Serenus' treatise on the sections of cones and cylinders was added to the mutilated *Conics* of Apollonius consisting of four Books only, the tract *On Polygonal Numbers* was added to the *Arithmetica* to make the latter a more complete volume. The topic of polygonal numbers was of great interest to the Pythagoreans. The beginning of this tract is quite like the beginning of Book I of the *Arithmetica*, with several definitions and preliminary propositions. *On Polygonal Numbers* is differentiated from the *Arithmetica* by its use of geometric proofs. It is worth mentioning that some editions of the *Arithmetica* include *On Polygonal Numbers* as Book VIII.

Diophantus' most important work is the *Arithmetica*, which he wrote in thirteen books, but of which only six have survived. Curiously, there are several names by which this text is known. The anonymous author of prolegomena to Nicomachus' *Introductio Arithmetica* quotes it as Diophantus' "thirteen Books of Arithmetic." A scholium on Iamblichus makes reference to "the last theorem of the first Book of Diophantus' Elements of Arithmetic." Yet another scholium on one of the epigrams in Metrodorus' collection speaks of the "Elements of Diophantus." This text is not a treatise on theoretical arithmetic, but rather deals with the computational arithmetic used in the solution of problems. It is a collection of problems, 189 to be precise, leading to determinate equations of the first degree such as those in Book I, and advancing in later books to those that lead to indeterminate equations of the second degree.

Arithmetica consists of more than 50 different classes of problems. Book I is restricted to determinate algebraic equations, Books II to V contain indeterminate problems, wherein expressions of the first or second degree in two or more variables are to be made squares or cubes, and Book VI deals with right-angled triangles purely arithmetically, in which some function of the sides is to be made a square or cube. The objective of Diophantus in *Arithmetica* is to obtain a numerical solution to each of his problems. He looks only for positive rational solutions and does not consider negative or irrational solutions. Surprisingly, whereas he has a general method of solving determinate equations of the second degree, he does not account for two roots, even when they are positive rational numbers. But it is Diophantus' methods of solution which are intriguing. With a skillful choice of unknowns and much ingenuity, Diophantus is able to simplify his algebraic manipulations to arrive at solutions to his problems. As the mathematical historian Hankel has best put it, "...it is difficult for a modern mathematician even after studying 100 Diophantine solutions to solve the 101^{st} problem...we shall be astonished to see how suddenly he leaves the broad high-road, dashes into a side-path and with a quick turn reaches the goal...Diophantus dazzles more than he delights. He is in a wonderful measure shrewd, clever, quick-sighted, indefatigable..." Hence the mystique of *Arithmetica* and the status of Diophantus as a master in the field of indeterminate analysis.

3 Diophantus' Notation

It is of significant interest to introduce the reader to the notation used by Diophantus. The first thing to note is that the notions of algebra were not formalized during Diophantus' time, so he did not use symbols such as x as we do today. For an unknown quantity, what we normally denote by x, Diophantus denoted by the Greek letter ς with an accent, thus $\acute{\varsigma}$. For denoting the coefficient of an unknown quantity, the Greek numeral is placed immediately after the unknown. So $\acute{\varsigma}\,\overline{\beta}$ represents $2x$. The square of an unknown quantity is represented by $\triangle^Y$. Similarly, x^3, x^4, x^5 and x^6 are represented by K^Y, $\triangle^Y\triangle$, $\triangle K^Y$ and $K^Y K$.

We now turn to the symbols for addition and subtraction. For addition, Diophantus used no symbol at all; rather, he just juxtaposed the terms to be added together. To distinguish between absolute numbers and variables, Diophantus used the word "units" and denoted them by $\overset{\circ}{M}$. So $x^3 + 13x^2 + 5x + 2$ is represented by $K^Y\,\overline{\alpha}\,\triangle^Y\,\overline{\iota\gamma}\,\varsigma\,\overline{\epsilon}\,\overset{\circ}{M}\,\overline{\beta}$. For subtraction, he denoted a $-$ sign by $\pitchfork$, and to prevent confusion, he placed all the negative terms in an expression together after all the positive terms. For instance, to represent $x^3 - 5x^2 + 8x - 1$, he would write $K^Y\,\overline{\alpha}\,\varsigma\,\overline{\eta}\ \pitchfork\ \triangle^Y\,\overline{\epsilon}\,\overset{\circ}{M}\,\overline{\alpha}$.

Next come the reciprocals of powers. For this his notation was similar to that for the usual powers, except that he added an exponent $\varkappa$ to the power. So $\varsigma^{\varkappa}\,\overline{\eta}$ corresponds to $8/x$ and $\triangle^{Y\varkappa}\,\overline{\sigma\nu}$ represents $250/x^2$.

Lastly, we discuss briefly the notation used by Diophantus for representing fractions. In the various manuscripts we have of Diophantus' work, there are different forms of notation used. Tannery adopts the rare representation where the denominator is placed above the numerator. So he writes $\frac{\iota\varsigma}{\rho\kappa\alpha}$ to represent $\frac{121}{16}$. A different representation is of the form $\iota\epsilon^{\delta} = \frac{15}{4}$. Another form used is writing the numerator first and the denominator after it, marking the denominator with a submultiple sign. So $\overline{\gamma}\,\delta'$ would mean $\frac{3}{4}$. Diophantus sometimes put a word between the numerator and denominator: $\grave{\epsilon}\upsilon\,\mu o\rho\acute{\iota}\omega$. For example, in problem 22 of Book V, he wrote $\overline{\beta}\,.\,\overline{\epsilon\chi}\ \ \grave{\epsilon}\upsilon\,\mu o\rho\acute{\iota}\omega\ \ \overline{\rho\kappa\beta}\,.\,\overline{\alpha\kappa\epsilon}$ to represent $25600/1221025$. He used this same notation when dealing with expressions involving unknown variables also.

4 A Goldmine To Treasure

With the biographical sketch of Diophantus, some knowledge of his texts, and his notation as our arsenal, we are now ready to use this machinery to appreciate the plethora of problems he posed and solved in *Arithmetica*. In the subsections to follow, we introduce the reader to a few of the gems this text contains.

4.1 Book I, Problem III

We shall first present here Simon Stevin's demonstration of the problem. Stevin translated and explained the first four books of *Arithmetica* in French. We provide our English translation here. Following that will be Heath's version of the same problem. It is a curious fact that Heath has a different statement of the problem altogether, and thus arrives at a different solution. This is valid for later problems where there are several solutions, and Diophantus sought only one; however, it leaves us with the lingering question as to what Diophantus' original statement and solution were.

Stevin's problem reads: "Divide 10 into two parts such that the larger is three times and two more than the smaller." The solution is as follows. We try as best to reproduce Stevin's interpretation of the solution, with slightly different notation for printing purposes.

Let the smaller required part be	Ⓢ
Then the larger is	3Ⓢ + 2
Their sum is	4Ⓢ + 2
Total equals	10

This reduces to 4Ⓢ equalling 8. And this gives Ⓢ equal to 2.
I say that 2 and 8 are the two required numbers.
To demonstrate, their sum, 2 and 8, is 10. Also, 8 is three
times and two more than 2, observe that this was required;
which was to be demonstrated.

This seems simple enough, and much like our modern algebraic manipulations.

We now present Heath's version: "To divide a given number into two numbers such that one is a given ratio of the other *plus* a given difference." His solution:

Given number 80, ratio 3 : 1, difference 4.

Lesser number x. Therefore the larger is $3x + 4$,

and $4x + 4 = 80$, so that $x = 19$.

The numbers are 61, 19.

Naturally, Heath uses contemporary notation such as the symbol x for the unknown, so as to make it easier to understand the material.

4.2 Book I, Problem XXVIII

This problem is posed as: "To find two numbers such that their sum and the sum of their squares are given numbers." We shall first present the solution given by Maximus Planudes in Greek, giving an English translation alongside. Then we shall show Heath's version.

Dioph. I.28

	Planudes		Modern Equivalent
	$\overline{\kappa}$	$\overline{\sigma\eta}$	[Given numbers] 20, 208
$\ddot{\epsilon}\kappa\theta.$	$\varsigma\overline{\alpha}\,\mu^{\circ}\overline{\iota}$	$\mu^{\circ}\overline{\iota}\ \pitchfork\ \varsigma\overline{\alpha}$	Put for the numbers $x + 10$, $10 - x$
$\tau\epsilon\tau\rho.$	$\triangle^{Y}\overline{\alpha}\varsigma\varsigma\overline{\kappa}\mu^{\circ}\overline{\rho}$	$\triangle^{Y}\mu^{\circ}\overline{\rho}\pitchfork\varsigma\varsigma\overline{\kappa}$	Squaring, we have $x^2 + 20x + 100$, $x^2 + 100 - 20x$
$\sigma\acute{\upsilon}\nu\theta.$	$\triangle^{Y}\overline{\beta}\mu^{\circ}\overline{\sigma}$	$\iota^{\sigma}\ \ \mu\overline{\sigma\eta}$	Adding, $2x^2 + 200 = 208$
$\grave{\alpha}\phi.$	$\triangle^{Y}\overline{\beta}$	$\iota^{\sigma}\ \ \mu\overline{\eta}$	Subtracting, $2x^2 = 8$
$\mu\epsilon\rho.$	$\triangle^{Y}\overline{\alpha}$	$\iota^{\sigma}\ \ \mu\overline{\delta}$	Dividing, $x^2 = 4$
	$\varsigma\overline{\alpha}$	$\iota^{\sigma}\ \ \mu\overline{\beta}$	$x = 2$
$\ddot{\upsilon}\pi.$	$\mu^{\circ}\overline{\iota\beta}$	$\mu^{\circ}\overline{\eta}$	Result: [the numbers are] 12, 8

Heath solves the problem by first stating a necessary condition, which he does not explain.

> *Necessary condition.* Double the sum of their squares must exceed the square of their sum by a square.

Given sum 20, given sum of squares 208.
Difference $2x$.
Therefore the numbers are $10+x$, $10-x$.

Thus $200+2x^2=208$, and $x=2$.

The required numbers are 12, 8.

As we can see, the solution is pretty much identical to the one Planudes attributes to Diophantus. We explain the necessary condition here in our words. Suppose the two numbers are a and b. Then what Heath says amounts to

$$2\left(a^2+b^2\right)-(a+b)^2=\square, \tag{1}$$

where $\square$ represents a square. But observe that the expression in (1) simplifies to $a^2+b^2-2ab=(a-b)^2$, which is always a square. So this raises a couple of interesting historical questions: Does Heath introduce this well-known identity as part of "authorial intrusion", and if so, why does he do so? Or was this identity one of many that Diophantus scattered throughout *Arithmetica*? The answers to these questions will certainly help in tracing the origin of this identity, though we do know that the identity $(a+b)^2=a^2+2ab+b^2$ can be located in its geometric form as early as in Book II of Euclid's *Elements*.

It is, however, the elegance of Diophantus' solution that is to be appreciated here. By a subtle choice of his two numbers, he was able to restrict his manipulations to involve only one variable, which led to a simple second degree equation, namely, $2x^2=8$. We are certainly in awe of this solution, for our approach to this problem would involve two variables: Let the numbers be x and y, so that $x+y=20$ and $x^2+y^2=208$. Since $y=20-x$, we get $x^2+(20-x)^2=208 \Rightarrow 2x^2-40x+192=0 \Rightarrow (x-12)(x-8)=0$, so $x=12$ or 8. This solution is mechanical and lengthy, to say the least. It also alludes to our earlier remark on Diophantus not considering the two roots of a quadratic equation. This solution lets him "get away with it" because the two solutions are the required numbers, so knowing one value gives the other one.

4.3 Book II, Problem I

Heath mentions that this problem, along with problems II through V, are mere repetitions of problems in Book I. Hence we shall present our translation of Stevin's solution.

The problem is posed as: "Find two numbers such that the sum of their squares is ten times their sum." For his solution, Diophantus made the additional assumption that one number is double the other. So letting the smaller number be x, the larger is then $2x$. Their sum is $3x$, and the sum of their squares is $x^2+4x^2=5x^2$. The given restriction implies $5x^2=30x$. Ignoring $x=0$ as a solution, we have $x=6$ and $2x=12$. To verify the result, observe that $6+12=18$ and $6^2+12^2=36+144=180=10\cdot 18$.

4.4 Book III, Problem II

This is an example of a problem whose solution seems arbitrary at first sight. The statement of the problem is: "To find three numbers such that the square of the sum of all three added to any one of them gives a square." Diophantus solved the problem in this manner: Let the square of the sum of all three be x^2, and the numbers $3x^2$, $8x^2$, $15x^2$. Hence $26x^2 = x$, so $x = \frac{1}{26}$, and the numbers are $\frac{3}{676}$, $\frac{8}{676}$, $\frac{15}{676}$.

On closer inspection, the solution makes logical sense. Diophantus chose the coefficients 3, 8, 15 for a reason: each of them is one less than a perfect square. This also explains why he chose 1 as the coefficient of the square of the sum of the three numbers. So $(3+8+15)\,x^2 = 26x^2 = \sqrt{x^2} = x$, which leads to the solution he obtained. As a check, we note that

$$\begin{aligned} \frac{3}{676}+\frac{1}{676} &= \frac{4}{676}=\left(\frac{2}{26}\right)^2, \\ \frac{8}{676}+\frac{1}{676} &= \frac{9}{676}=\left(\frac{3}{26}\right)^2, \\ \frac{15}{676}+\frac{1}{676} &= \frac{16}{676}=\left(\frac{4}{26}\right)^2. \end{aligned}$$

Once again, Diophantus excelled as a master at choosing his unknowns very deftly. It seems plausible that he kept his ultimate solutions in mind, which then helped him select the coefficients of his variable quantities. Nevertheless, we remain truly impressed with his algebraic prowess and insight.

4.5 Book IV, Problem I

The solution to this problem uses an approach similar to that in I.28, which is why we discuss this problem. It verifies the fact (and the Pigeonhole Principle) that although the *Arithmetica* contains some 50 classes of problems and solutions, some of the problems belong to the same category.

The problem reads: "To divide a given number into two cubes such that the sum of their sides is a given number." Here, the given number is 370, and the given sum of sides is 10. Diophantus let the sides be $5+x$ and $5-x$. Then

$$\begin{aligned} (5+x)^3+(5-x)^3 &= (5+x+5-x)\left[(5+x)^2-(5+x)(5-x)+(5-x)^2\right] \\ &= 10\left(50+2x^2-25+x^2\right) \\ &= 30x^2+250 \\ &= 370, \end{aligned}$$

so $30x^2 = 120 \Rightarrow x = 2$, and the required cubes are $7^3 = 343$ and $3^3 = 27$.

4.6 Book V, Problem V

This problem is of interest because its solution makes reference to a result in *Porismata*. In our opinion, this serves as further evidence for Heath's view that *Porismata* was indeed an independent text by Diophantus, and not a part of *Arithmetica*.

The problem is "to find three squares such that the product of any two added to the sum of those two, or to the remaining square, gives a square." Diophantus remarked at the onset: "We have it in the *Porisms* that, if the squares on any two consecutive numbers be taken, and a third number be also taken which exceeds twice the sum of the squares by 2, we have three numbers such that the product of any two added to those two or to the remaining number gives a square." With this, he proceeds to solve the problem: Assume as the first square $x^2 + 2x + 1$, and as the second $x^2 + 4x + 4$, so that the third number is $4x^2 + 12x + 12$. Therefore $x^2 + 3x + 3$ is a square, say $(x - 3)^2$, and $x = \frac{2}{3}$. Therefore $\left(\frac{25}{9}, \frac{64}{9}, \frac{196}{9}\right)$ is a solution. It is easy to check that

$$\begin{aligned}
\frac{25}{9} \cdot \frac{64}{9} + \frac{25}{9} + \frac{64}{9} &= \frac{2401}{81} = \left(\frac{49}{9}\right)^2 \\
\frac{25}{9} \cdot \frac{196}{9} + \frac{25}{9} + \frac{196}{9} &= \frac{6889}{81} = \left(\frac{83}{9}\right)^2 \\
\frac{64}{9} \cdot \frac{196}{9} + \frac{64}{9} + \frac{196}{9} &= \frac{14884}{81} = \left(\frac{122}{9}\right)^2 \\
\frac{25}{9} \cdot \frac{64}{9} + \frac{196}{9} &= \frac{3364}{81} = \left(\frac{58}{9}\right)^2 \\
\frac{25}{9} \cdot \frac{196}{9} + \frac{64}{9} &= \frac{5476}{81} = \left(\frac{74}{9}\right)^2 \\
\frac{64}{9} \cdot \frac{196}{9} + \frac{25}{9} &= \frac{12769}{81} = \left(\frac{113}{9}\right)^2
\end{aligned}$$

It is not hard to verify this algebraically also. Using the result quoted from *Porismata*, let the three squares be $x = m^2$, $y = (m + 1)^2$, and $z = 4\left(m^2 + m + 1\right)$. Then

$$\begin{aligned}
xy + x + y &= \left(m^2 + m + 1\right)^2 && (2) \\
yz + y + z &= \left(2m^2 + 3m + 3\right)^2 && (3) \\
zx + z + x &= \left(2m^2 + m + 2\right)^2 && (4) \\
xy + z &= \left(m^2 + m + 2\right)^2 && (5) \\
yz + x &= \left(2m^2 + 3m + 2\right)^2 && (6) \\
zx + y &= \left(2m^2 + m + 1\right)^2 && (7)
\end{aligned}$$

Once again, we admire the cleverness employed by Diophantus in *Porismata* in choosing x, y, z so as to produce the remarkable identities (2) through (7).

4.7 Book VI, Problem XXIV

As mentioned earlier, Book VI deals with right-angled triangles, where some linear or quadratic function of the sides is to be made a square or a cube. This book contains several interesting, and rather difficult, problems. We shall present only one of the easier problems, which asks "to find a right-angled triangle such that one perpendicular is a cube, the other is the difference between a cube and its side, and the hypotenuse is the sum of a cube and its side."

Diophantus chose the hypotenuse to be $x^3 + x$, and one perpendicular to be $x^3 - x$. Then by Pythagoras' Theorem, the other perpendicular is $\sqrt{(x^3+x)^2 - (x^3-x)^2} = 2x^2$. Since $2x^2$ is to equal a cube, he lets $2x^2 = x^3$. Then $x = 2$, and the triangle is the $6 - 8 - 10$ right-angled triangle. As a check, $6 = 2^3 - 2$, $8 = 2^3$, and $10 = 2^3 + 2$.

4.8 A Glimpse At The Congruent Number Problem

A number is said to be congruent if it is the area of a right-angled triangle with rational sides. For example, 5 is a congruent number because it is the area of the $\frac{3}{2} - \frac{20}{3} - \frac{41}{6}$ right-angled triangle. It was Euler who coined the name "congruent number" and asked for a characterization of all congruent numbers. However, the problem can be traced back much earlier. Taking a lead from the nature of problems in Book VI of the *Arithmetica*, Bachet wrote an appendix of problems on right-angled triangles. Problem 20 in this appendix is "to find a right-angled triangle such that its area is equal to a given number." This led to Pierre de Fermat's remarkable discovery that the area of a right-angled triangle with rational sides cannot be a square number. But this is only a partial solution of the congruent number problem. In 1983, Jerrold Tunnell recast the problem into one involving elliptic curves by considering the rational points on the elliptic curves $y^2 = x^3 - n^2x$, thus making some progress towards a solution of the problem. The congruent number problem thus remains an easy-to-state but as yet unsolved problem.

5 Conclusion

From the few problems presented above, we hope the reader is convinced of the variety and breadth of Diophantus' problems, and the elegance and ingenuity of his solutions. He contributed several new theorems to number theory, but the sheer influence of the *Arithmetica* on modern Mathematics is remarkable enough. He also introduced symbolic notation for his algebraic manipulations, much of which is used today in essence, if not in actual form. It is naturally impossible to reproduce even a fraction of his work in such a paper, but we certainly hope we have done justice to the mathematical significance of the work of Diophantus.

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