Purdue Problem of the Week

Gagan Tara Nanda
Senior at University of California, Berkeley
1927 Dwight Way, #102, Berkeley, CA 94704-1945

06^{th} November, 2003

1. We list the first few Lucas numbers, their squares, and the first few values of $\sum_{k=0}^{n} L_k^2$:

$$
\begin{array}{ll}
2, 1, 3, 4, 7, 11, 18 & \text{(First 7 Lucas numbers)} \\
4, 1, 9, 16, 49, 121 & \text{(Squares of first 6 Lucas numbers)} \\
4, 5, 14, 30, 79, 200 & (\text{Value of } \sum_{k=0}^{n} L_k^2 \text{ for } n = 0, 1, 2, 3, 4, 5)
\end{array}
$$

We now observe that

$$
\begin{aligned}
4 &= 2 \cdot 1 + 2; \\
5 &= 1 \cdot 3 + 2; \\
14 &= 3 \cdot 4 + 2; \\
30 &= 4 \cdot 7 + 2; \\
79 &= 7 \cdot 11 + 2; \\
190 &= 11 \cdot 18 + 2.
\end{aligned}
$$

From here we conjecture that

$$\sum_{k=0}^{n} L_k^2 = L_n L_{n+1} + 2.$$

We shall prove this by induction. The base case clearly holds, since

$$\sum_{k=0}^{0} L_k^2 = 4 = 2 \cdot 1 + 2 = L_0 L_1 + 2.$$

Assume the result is true for $n = m$, that is,

$$\sum_{k=0}^{m} L_k^2 = L_m L_{m+1} + 2.$$

Then for $n = m + 1$, we have

$$\begin{aligned}\sum_{k=0}^{m+1} L_k^2 &= \sum_{k=0}^{m} L_k^2 + L_{m+1}^2 \\ &= L_m L_{m+1} + 2 + L_{m+1}^2 \\ &= L_{m+1}\left(L_m + L_{m+1}\right) + 2 \qquad \text{(By the induction hypothesis)} \\ &= L_{m+1} L_{m+2} + 2.\end{aligned}$$

Hence the result is true for all non-negative integers n, and we conclude that

$$\sum_{k=0}^{n} L_k^2 = L_n L_{n+1} + 2.$$