Purdue Problem of the Week

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11. Let the points of the triangle be $A\,(x_1, y_1)$, $B\,(x_2, y_2)$, and $C\,(x, y)$, where C lies on K. Let the center of K be $O\,(c, d)$, so that $(x-c)^2 + (y-d)^2 = 1$. Then the centroid of $\triangle ABC$ is the point

$$W = (e, f) = \left(\frac{x_1 + x_2 + x}{3}, \frac{y_1 + y_2 + y}{3}\right).$$

We can now find the locus of W as C varies:

$$\begin{aligned}\left(e - \frac{x_1 + x_2}{3} - \frac{c}{3}\right)^2 + \left(f - \frac{y_1 + y_2}{3} - \frac{d}{3}\right)^2 &= \left(\frac{x-c}{3}\right)^2 + \left(\frac{y-d}{3}\right)^2 \\ &= \frac{(x-c)^2 + (y-d)^2}{9} \\ &= \frac{1}{9}.\end{aligned}$$

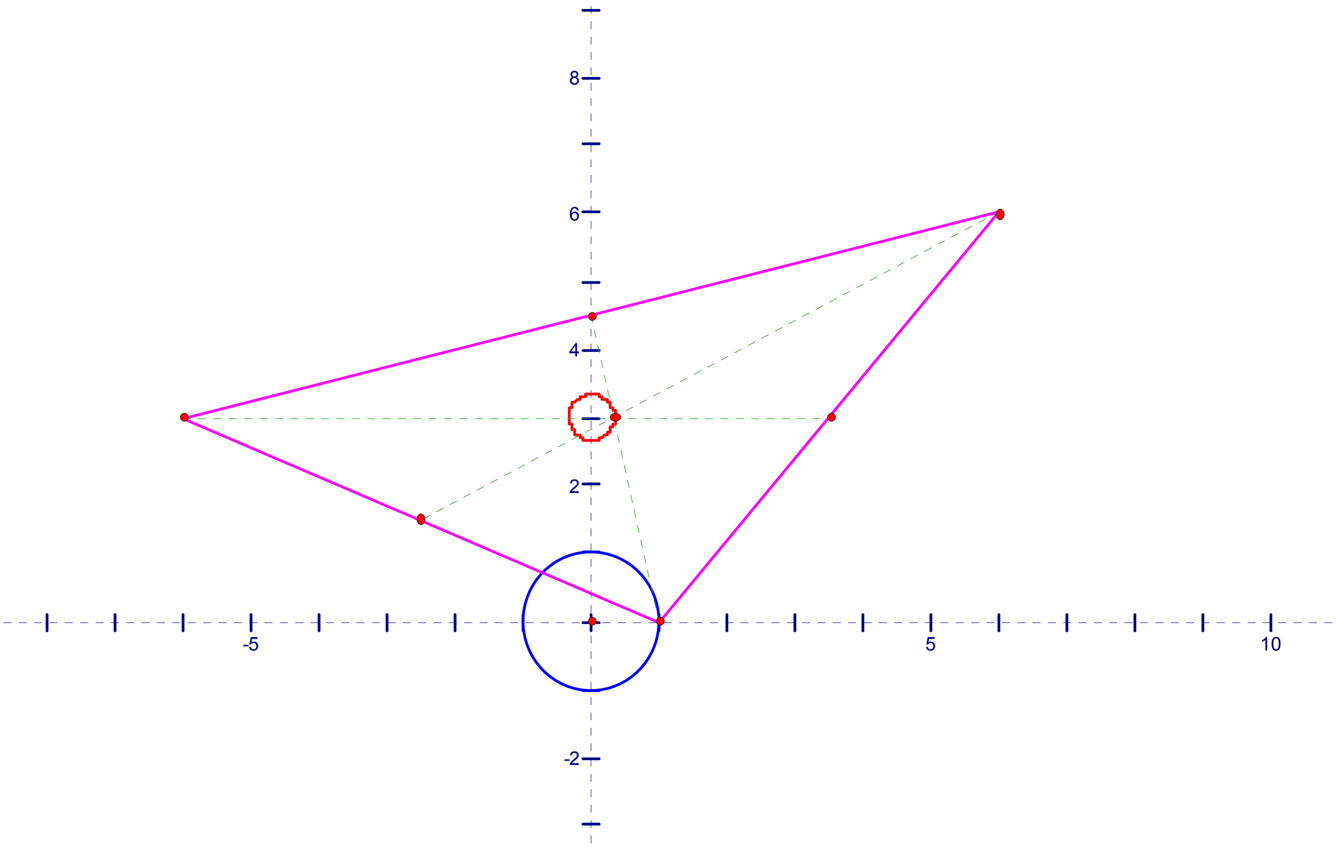
Hence the locus of W is a circle, with center

$$O' = \left(\frac{x_1 + x_2 + c}{3}, \frac{y_1 + y_2 + d}{3}\right) = \frac{O + A + B}{3} \qquad \text{(In vector notation.)}$$

and radius

$$r' = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

More generally, if K has radius r, then $r' = \frac{r}{3}$. We illustrate this with a diagram for the case $O = (0, 0)$, $A = (-6, 3)$, $B = (6, 6)$, and $r = 1$.



I believe this result can be generalized to ellipsis and hyperbolae. Consider any such conic K represented by

$$K : b^2 (x-c)^2 \pm a^2 (x-d)^2 = a^2 b^2. \tag{1}$$

Then the centroid of $\triangle ABC$ is still the point

$$W = (e, f) = \left(\frac{x_1 + x_2 + x}{3}, \frac{y_1 + y_2 + y}{3} \right).$$

Observe that the locus of W as C varies is given by:

$$\begin{aligned} b^2 \left(e - \frac{x_1 + x_2}{3} - \frac{c}{3} \right)^2 \pm a^2 \left(f - \frac{y_1 + y_2}{3} - \frac{d}{3} \right)^2 &= b^2 \left(\frac{x-c}{3} \right)^2 \pm a^2 \left(\frac{y-d}{3} \right)^2 \\ &= \frac{b^2 (x-c)^2 \pm a^2 (y-d)^2}{9} \\ &= \frac{a^2 b^2}{9}, \end{aligned}$$

which represents the same type of conic as K, with area $\frac{1}{9}$ that of K.