Note Title

Fourier Analysis

- Submitted a paper in 1807 to Academy of Sciences of Paris

- The paper was rejected for lack of rigour but contained the ideas for what has become:

  Fourier Analysis

Acording to Fourier if

$F(t)$ is a periodic function

T = period

Then

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega_T t + b_n \sin \omega_T t \right)$$

where

$$\omega_T = \frac{2\pi}{T} \qquad a_n = \frac{2}{T}\int_0^T F(t) \cos n\omega_T t \, dt \quad n = 1, 2$$

$$a_0 = \frac{2}{T}\int_0^T F(t)\, dt \qquad b_n = \frac{2}{T}\int_0^T F(t) \sin n\omega_T t \, dt$$

Note that

$$a_0 = \text{twice average of the function of period } T$$

A line graph of the amplitudes of the Fourier series components is called a <u>frequency spectrum</u>

$f_0 = \frac{1}{T}$ fundamental or first harmonic

$f_n = n f_0$ higher harmonics

Fourier Series Approximation

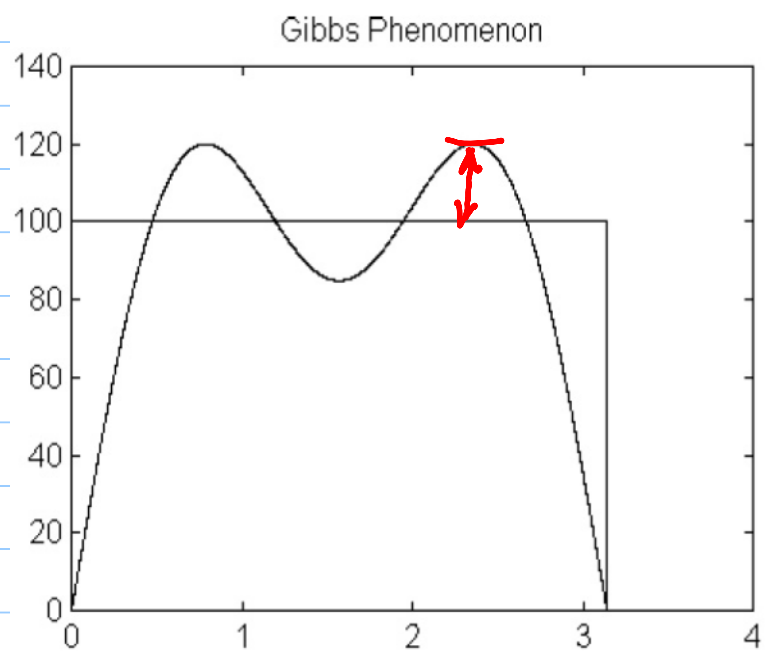
$$x(t) = a_0 + a_1 \cos(\omega_0 t + \theta_1) + a_2 \cos(2\omega_0 t + \theta_2)$$

$$+ \cdots a_n \cos(N\omega_0 t + \theta_N)$$

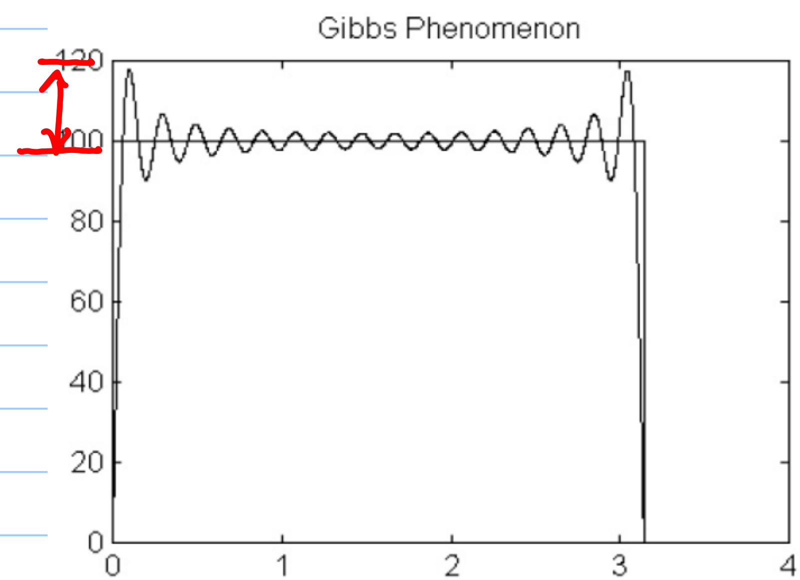
$$\omega_0 = \frac{2\pi}{T}$$

$$0 \le \theta_1 \ldots \theta_N < 2\pi$$

Gibbs Phenomenon



2 terms



16 terms

Complex representation of Fourier series

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

Bertrand Russell

"the most beautiful, profound and subtle expression in mathematics"

Remember

$$f(t) = a_0 + \sum a_n \sin(\omega_n t) + \sum b_n \cos \omega_n t$$

Where a_n + b_n are as previosly defined

Since $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Substituting

$$f(t) = a_0 + \sum \frac{a_n}{2}\left(e^{jn\omega t} + e^{-jn\omega t}\right) + \frac{b_n}{2i}\left(e^{jn\omega t} - e^{-jn\omega t}\right)$$

Following more algebra we get shortened form of Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

where

$$C_n = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-jn\omega t} \, dt$$

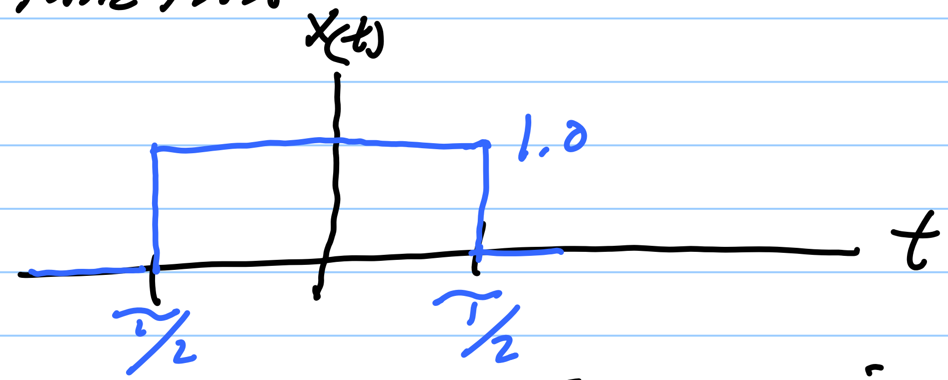
We can take the sum to an integral and we get the Fourier transform

$$\underset{\sim}{X}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt$$

and an inverse transform

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underset{\sim}{X}(\omega)\, e^{i\omega t}\, d\omega$$

Consider function

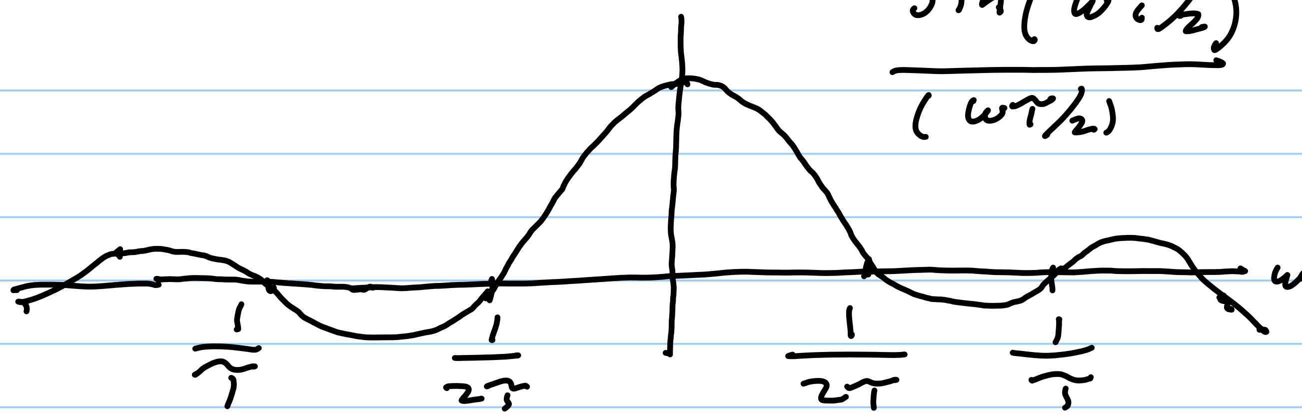


$$\underset{\sim}{X}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-j\omega} \left(e^{-j\omega t} \right) \right|_{-\tau/2}^{\tau/2}$$

$$= \frac{\tau}{\sqrt{2\pi}} \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$$

$$\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$$



Properties

$f(t) \Longleftrightarrow F(\omega)$ fourier tranform Pair

Time shift

$$f(t-t_0) \Longleftrightarrow e^{-i\omega t_0} F(\omega)$$

Convolution

$$\int f_1(\tau) f_2(t-\tau)\, d\tau \Longleftrightarrow F_1(\omega) F_2(\omega)$$

Time Reversal

$$f(-t) = F(-\omega)$$

Differentiation

$$f'(t) = j\omega F(\omega)$$

Laplace Transform

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Fourier Transform

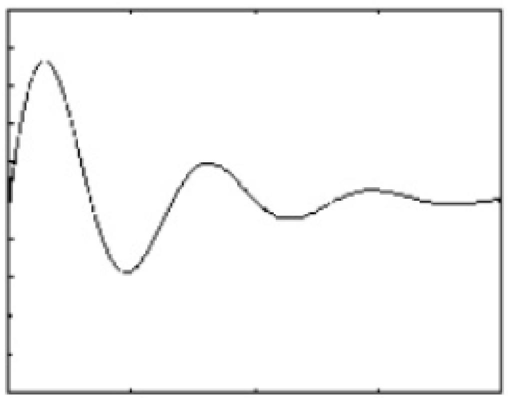
$$F(\omega) = \int_{-\infty}^\infty f(t)\, e^{-j\omega}\, dt$$

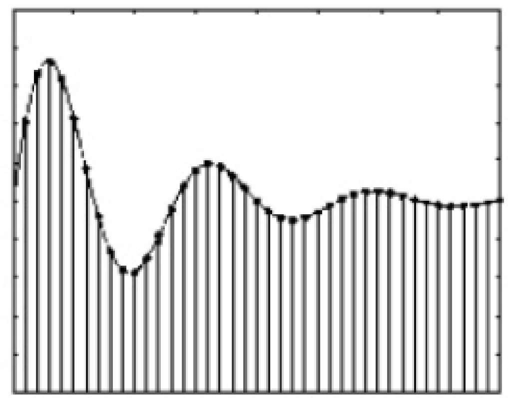
Very similar

$$f'(t) \Rightarrow s F(s)$$

$$f'(t) \Rightarrow j\omega F(s)$$

Sampling

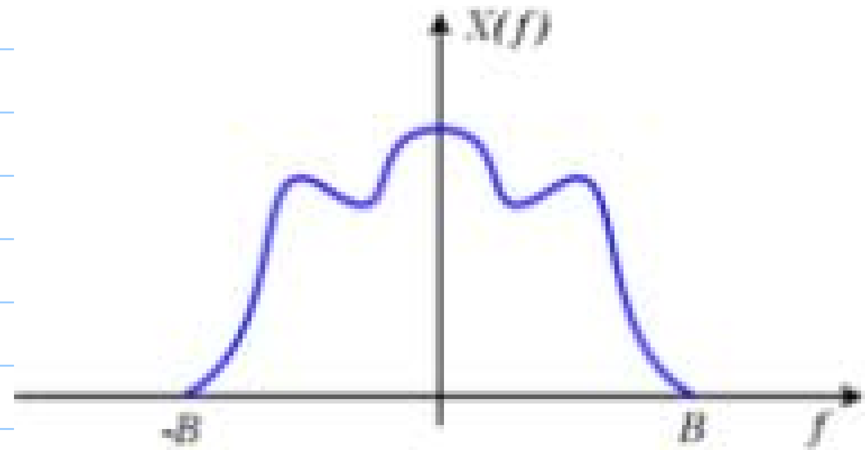




If Δ is the time between samples

$$h_n = h(n\Delta n) \quad \text{where } n = 0, \pm 1, \pm 2 \ldots$$

Consider a bandwidth limited functions



Nyquist-Shannon Sampling Theorem

Let

$x(t)$ = continuous-time signal

$X(f)$ = fourier transform

Then

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi i f t}\, dt$$

Bandwidth limited

$$X(f) = 0 \quad \text{for } |f| > B$$

Condition for exact reconstructability

$$f_s > 2B$$

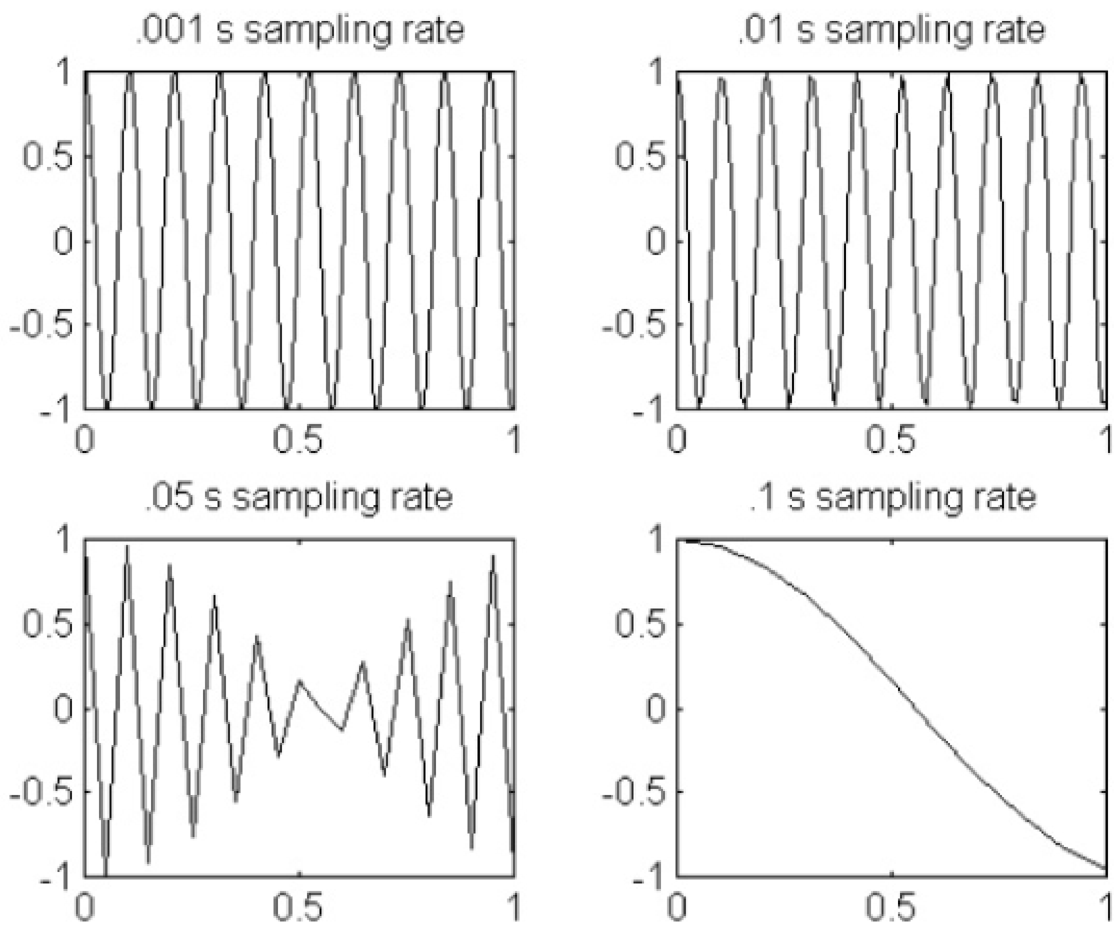
2B is the Nyquist Rate

Sampling interval

$$T_s = \frac{1}{f_s}$$

$$T < \frac{1}{2B}$$ Nyquist interval

Cosine Sampling



Moiré

Properly sampled image

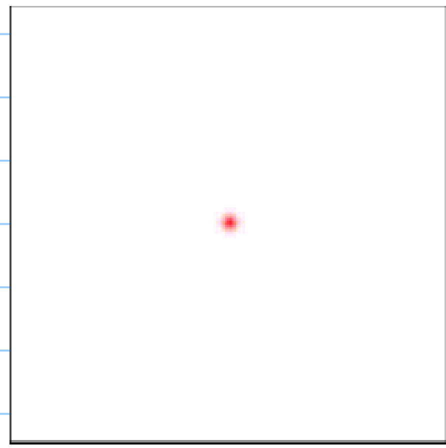
Subsampled image

Anti-aliasing filter

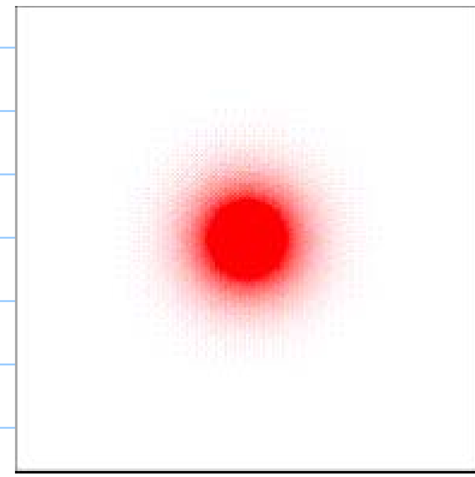
- use an optical low pass filter or blur filter

- Sound Blaster card requires a low pass filter in a data aquisition circuit when connecting to a sound card to ensure no frequencies > half the sampling rate

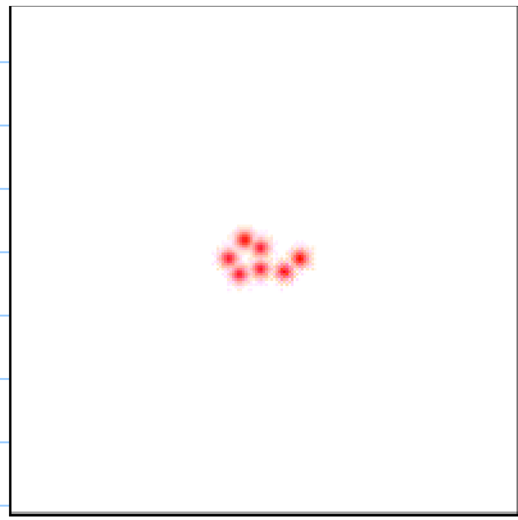
2 dimensional Fourier transforms



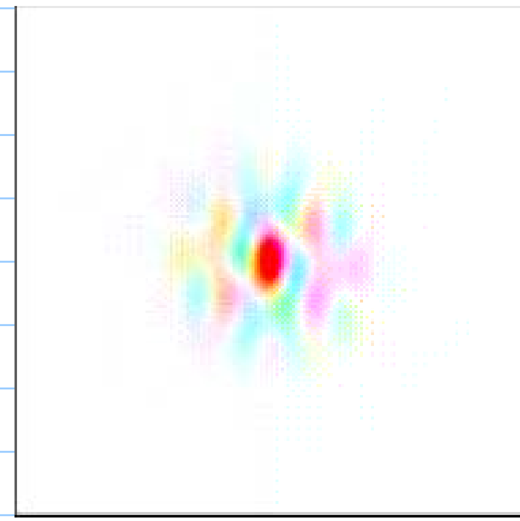
atom



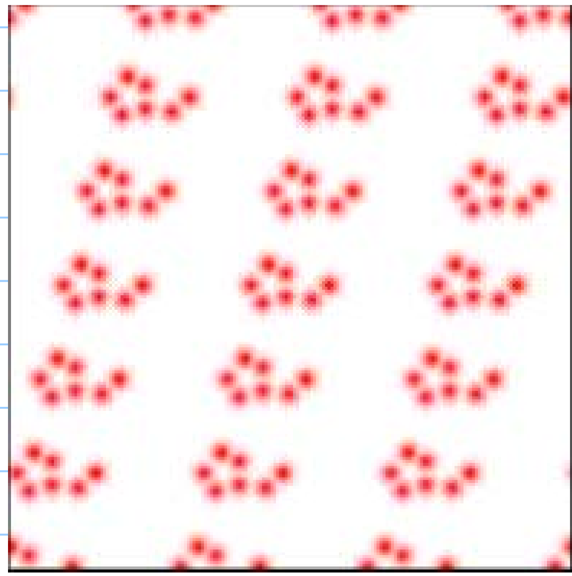
atom fourier tranform



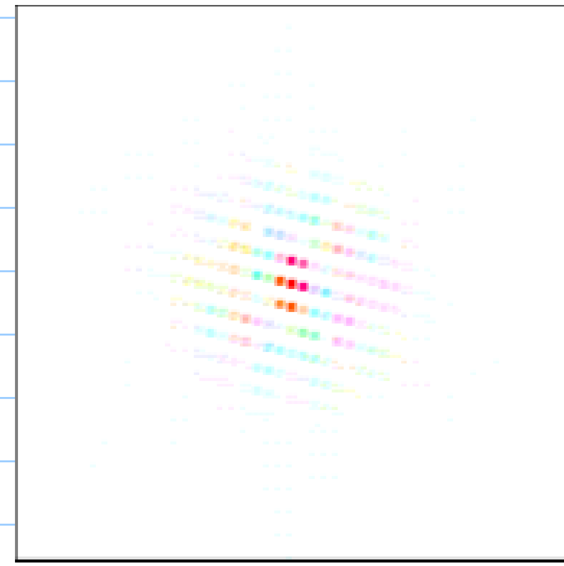
molecule



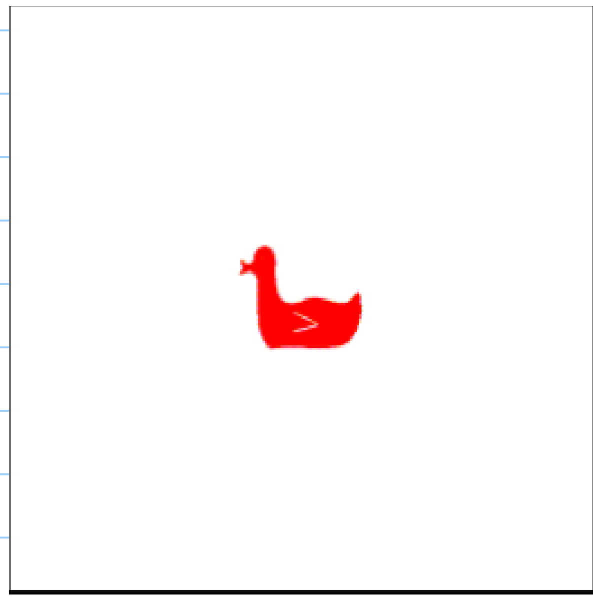
molecule fourier transform



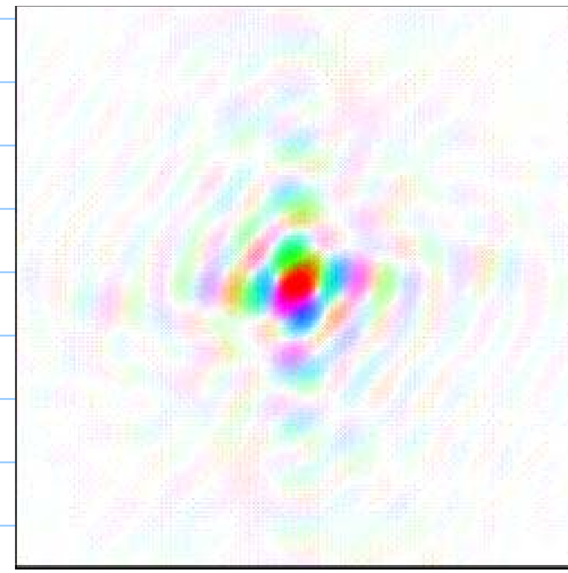
Crystal



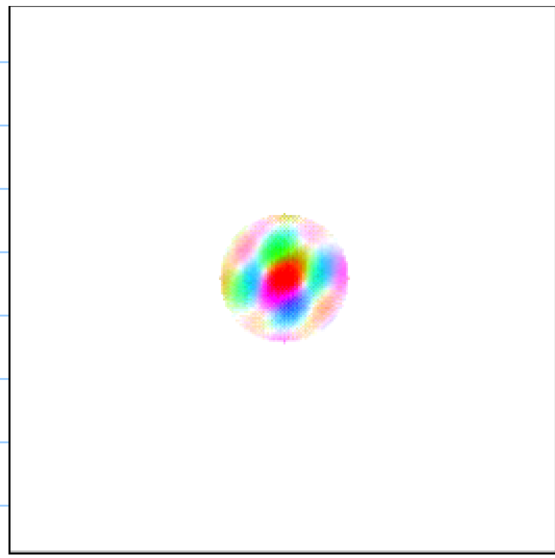
Crystal fourier transform



duck

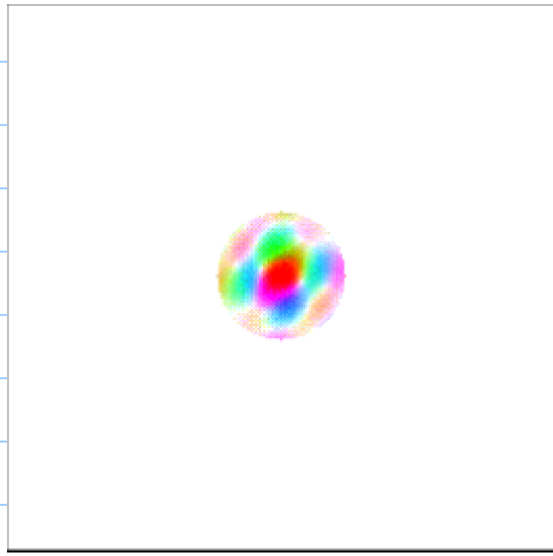


duck fourier transform



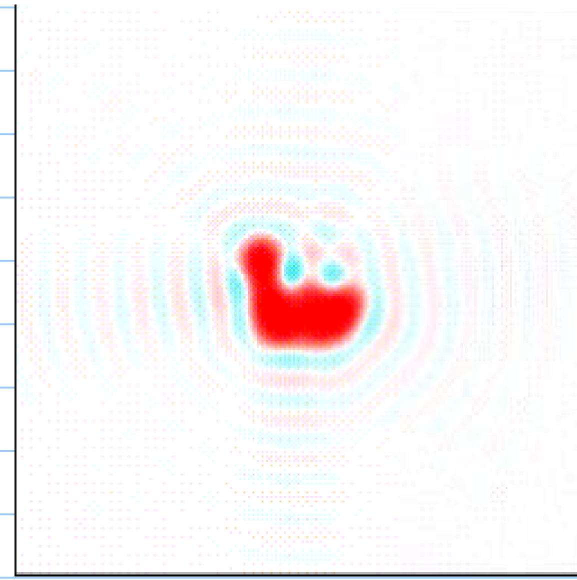
Reduced resolution
fourier transform of duck

FT



Reduced resolution
fourier transform of duck

Inverse Fourier Transform



Low resolution
Duck