Section 10.2

<u>Theorem</u>: If a smooth curve C is given parametrically by $x = f(t)$, $y = g(t)$, then the slope dy/dx of the tangent line to C at $P(x, y)$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $\frac{dx}{dt} \neq 0$.

Exercise 1. Let C be the curve with parametrization

$$x = 2t,\ y = t^2 - 1; \quad -1 \leq t \leq 2.$$

Find the slopes of the tangent line and normal line to C at $P(x, y)$.

Exercise 2. Let C be the curve with parametrization

$$x = t^3 - 3t,\ y = t^2 - 5t - 1; -\infty < t < \infty,$$

(a) Find an equation of the tangent line to C at the point corresponding to $t = 2$.
(b) For what values of t is the tangent line horizontal or vertical?

Class Exercise 1. Find dy/dx if $x = 1/t$ and $y = \sqrt{t}e^{-t}$.

Class Exercise 2. Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.
(a) $x = t - t^{-1}$, $y = 1 + t^2$; $t = 1$ (b) $x = \sin^3\theta$, $y = \cos^3\theta$; $\theta = \pi/6$

If a curve C is parametrized by $x = f(t)$, $y = g(t)$, and if y' is a differentiable function of t, we can find d^2y/dx^2 by applying the above theorem to y' as follows.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$

It is important to observe that $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$.

Exercise 3. Let C be the curve with parametrization

$$x = e^{-t},\ y = e^{2t},\ -\infty < t < \infty.$$

(a) Sketch the graph of C and indicate the orientation.
(b) Use the above theorems to find dy/dx and d^2y/dx^2.
(c) Find a function k that has the same graph as C, and use $k'(x)$ and $k''(x)$ to check the answers.
(d) Discuss the concavity of C.

Class Exercise 3. Find (a) dy/dx and (b) d^2y/dx^2 in terms of t.
(i) $x = \cos t$, $y = \sqrt{3}\cos t$ (ii) $x = 1/t$ and $y = -2 + \ln t$ (iii) $x = t^2 + t$, $y = t^2 - t$

<u>Theorem</u>: If a smooth curve C is given parametrically by $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, and if C does not intersect itself, except possibly for $t = a$ and $t = b$, then the length L of C is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\, dt.$$

Exercise 4. Find the length of one arch of the cycloid that has the parametrization

$$x = t - \sin t,\ y = 1 - \cos t; -\infty < t < \infty$$

Class Exercise 4. Find the length of the curve.
(a) $x = \cos t$, $y = t + \sin t$, $0 \leq t \leq \pi$ (b) $x = \frac{(2t+3)^{3/2}}{3}$, $y = t + \frac{t^2}{2}$, $0 \leq t \leq 3$
(c) $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $0 \leq t \leq 1$ (d) $x = 8\cos t + 8t\sin t$, $y = 8\sin t - 8t\cos t$, $0 \leq t \leq \pi/2$
(e) $x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t$, $0 \leq t \leq \pi/3$ (f) $x = e^t - t^2$, $y = t + e^{-t}$, $-1 \leq t \leq 2$

<u>Surface Area Formula</u>: If the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the x-axis, where f, g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by $S = \int_\alpha^\beta 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$.

Exercise 5. Verify that the surface area of a sphere of radius a is $4\pi a^2$.

Class Exercise 5. Find the area of the surface generated by revolving the curve about the indicated axis. (a) $x = \cos t$, $y = 2 + \sin t$, $0 \leq t \leq 2\pi$; x-axis
(b) $x = (2/3)t^{3/2}$, $y = 2\sqrt{t}$, $0 \leq t \leq 2$; y-axis (c) $x = t + 1$, $y = t^2 + 2$, $0 \leq t \leq 3$; y-axis
(d) $x = \ln(\sec t + \tan t) - \sin t$, $y = \cos t$, $0 \leq t \leq \pi/3$; x-axis

Homework: 3-23 (every 4th), 31, 33, 45, 49, 67-73 ODD