## Section 10.4

**Theorem**: If $f$ is continuous and $f(\theta) \geq 0$ on $[\alpha, \beta]$, where $0 \leq \alpha < \beta \leq 2\pi$, then the area $A$ of the region bounded by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is

$$A = \int_\alpha^\beta \tfrac{1}{2}\,[f(\theta)]^2\, d\theta = \int_\alpha^\beta \tfrac{1}{2} r^2\, d\theta.$$

**Exercise 1.** Find the area of the region bounded by the cardioid $r = 2 + 2\cos\theta$.

**Exercise 2.** Find the area $A$ of the region $R$ that is inside the cardioid $r = 2 + 2\cos\theta$ and outside the region $r = 3$.

**Class Exercise 1.** Find the area of the region.
(a) inside the oval limacon $r = 4 + 2\cos\theta$
(b) inside the cardioid $r = a(1+\cos\theta)$, $a > 0$
(c) inside the lemniscate $r^2 = 2a^2\cos 2\theta$, $a > 0$
(d) inside one leaf of the four-leaved rose $r = \cos 2\theta$
(e) inside one loop of the lemniscate $r^2 = 4\sin 2\theta$
(f) inside the six-leaved rose $r^2 = 2\sin 3\theta$
(g) shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$
(h) shared by the circles $r = 1$ and $r = 2\sin\theta$

**Length of a Polar Curve**: If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as $\theta$ runs from $\alpha$ to $\beta$, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + (\tfrac{dr}{d\theta})^2}\, d\theta.$$

**Exercise 3.** Find the length of the cardioid $r = 1 - \cos\theta$.

**Class Exercise 2.** Find the length of the curve.
(a) the spiral $r = \theta^2$, $0 \leq \theta \leq \sqrt{5}$.
(b) the spiral $r = e^\theta/\sqrt{2}$, $0 \leq \theta \leq 2\pi$.
(c) the cardioid $r = 1 + \cos\theta$
(d) the curve $r = a\sin^2(\theta/2)$, $0 \leq \theta \leq \pi$, $a > 0$
(e) the parabolic segment $r = 6/(1+\cos\theta)$, $0 \leq \theta \leq \pi/2$
(f) the parabolic segment $r = 2/(1-\cos\theta)$, $\pi/2 \leq \theta \leq \pi$
(g) the curve $r = \cos^3(\theta/3)$, $0 \leq \theta \leq \pi/4$
(h) the curve $r = \sqrt{1+\sin 2\theta}$, $0 \leq \theta \leq \pi\sqrt{2}$

**Tests for symmetry**

(i) The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if substitution of $-\theta$ for $\theta$ leads to an equivalent equation.
(ii) The graph of $r = f(\theta)$ is symmetric with respect to the vertical line $\theta = \pi/2$ if substitution of either (a) $\pi - \theta$ for $\theta$ or (b) $-r$ for $r$ and $-\theta$ for $\theta$ leads to an equivalent equation.
(iii) The graph of $r = f(\theta)$ is symmetric with respect to the pole if substitution of either (a) $-r$ or $r$ or (b) $\pi + \theta$ leads to an equivalent equation.

**Theorem**: The slope $m$ of the tangent line to the graph of $r = f(\theta)$ at the point $P(r, \theta)$ is

$$m = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

**Exercise 7.** For the cardioid $r = 2 + 2\cos\theta$ with $0 \le \theta \le 2\pi$, find
(a) the slope of the tangent line at $\theta = \pi/6$ (b) the points at which the tangent line is horizontal
(c) the points at which the tangent line is vertical

**Exercise 8.** Find the horizontal and vertical tangents of the cardioid $r = 1 - \cos\theta$, $0 \le \theta \le 2\pi$.

**Class Exercise 3.** Find the slope of the curve at each indicated point.
(a) $r = \cos 2\theta$, $\theta = 0, \pm\pi/2, \pi$ (b) $r = 3(1 - \cos\theta)$, $\theta = \pi/3, 2\pi/3, \pi, 3\pi/2$

**Class Exercise 4.** Find equations for the horizontal and vertical tangent lines to the curve.
(a) $r = -1 + \sin\theta$, $0 \le \theta \le 2\pi$ (b) $r = 1 + \cos\theta$, $0 \le \theta \le 2\pi$
(c) $r = 2\sin\theta$, $0 \le \theta \le \pi$ (d) $r = 3 - 4\cos\theta$, $0 \le \theta \le 2\pi$

Homework: 1-41 (every 4th), 49, 51, 63-71 ODD

## Section 10.5

**Definition**: A **parabola** is the set of points in a plane that are equidistant from a fixed point $F$ (called the **focus**) and a fixed line (called the **directrix**).

**Equation**: An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py.$$

**Equation**: $y^2 = 4px$ is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$.

**Exercise 9.** Find the vertex, focus, and directrix of the $2y^2 = 5x$ and sketch its graph. (Section 10.5 #2)

**Class Exercise 5.** Find the vertex, focus, and directrix of the parabola $3x^2 + 8y = 0$ and sketch its graph. (Section 10.5 #4)

**Class Exercise 6.** Find the vertex, focus, and directrix of the parabola $(y - 2)^2 = 2x + 1$ and sketch its graph. (Section 10.5 #6)

**Class Exercise 7.** Find the vertex, focus, and directrix of the parabola $2x^2 - 16x - 3y + 38 = 0$ and sketch its graph. (Section 10.5 #8)

**Definition**: An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points $F_1$ and $F_2$ is a constant.

**Equation**: The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \ge b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

**Equation**: The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \ge b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

**Exercise 10.** Find the vertices and foci of the ellipse $\frac{x^2}{36} + \frac{y^2}{8} = 1$ and sketch its graph. (Section 10.5 #12)

**Class Exercise 8.** Find the vertices and foci of the ellipse $100x^2 + 36y^2 = 225$ and sketch its graph. (Section 10.5 #14)

**Class Exercise 9.** Find the vertices and foci of the ellipse $x^2 + 3y^2 + 2x - 12y + 10 = 0$ and sketch its graph. (Section 10.5 #16)

**Definition**: A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points $F_1$ and $F_2$ is a constant.

**Equation**: The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm\, c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm\,(b/a)x$.

**Equation**: The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$ and asymptotes $y = \pm\,(a/b)x$.

**Exercise 11.** Find the vertices, foci, and asymptotes of the hyperbola

$$\frac{x^2}{36} - \frac{y^2}{64} = 1$$

and sketch its graph.

**Class Exercise 10.** Find the vertices, foci, and asymptotes of the hyperbola $y^2 - 16x^2 = 16$ and sketch its graph. (Section 10.5 #22)

**Class Exercise 11.** Find the vertices, foci, and asymptotes of the hyperbola $9y^2 - 4x^2 - 36y - 8x = 4$ and sketch its graph. (Section 10.5 #24)

Homework: 1-21 (every 4th), 27-47 (every 4th)