Section 11.1

Definition: A **sequence** is a function f whose domain is the set of positive integers.

Exercise 1. List the first four terms and the tenth term of each sequence:

(a) $\{ \frac{n}{n+1} \}$
(b) $\{ 2 + (0.1)^n \}$
(c) $\{ (-1)^{n+1} \frac{n^2}{3n-1} \}$
(d) $\{4\}$

For some sequences we state the first term a_1, together with a rule for obtaining any term a_{k+1} from the preceding term a_k whenever $k \geq 1$. We call this a **recursive definition**, and the sequence is said to be defined **recursively**.

Exercise 2. Find the first four terms and the nth of the sequence defined recursively as follows:

$$a_1 = 3 \text{ and } a_{k+1} = 2a_k \text{ for } k \geq 1.$$

Class Exercise 1. List the first five terms of the sequence.
(a) $a_n = \frac{3^n}{1+2^n}$
(b) $a_n = \cos \frac{n\pi}{2}$
(c) $a_n = \frac{(-1)^n n}{n!+1}$
(d) $a_1 = 6$, $a_{n+1} = \frac{a_n}{n}$
(e) $a_1 = 2$, $a_2 = 1$, $a_{n+1} = a_n - a_{n-1}$

Exercise 3. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.
(a) $\{1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots\ldots \}$

Class Exercise 2. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.
(a) { 5, 8, 11, 14, 17 ...... }
(b) {1,0,-1,0,1,0,-1,0,....}

Definition: A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or it is **divergent**).

Definition: The notation

$$\lim_{n\to\infty} a_n = \infty$$

means that for every positive real number P there exists a number N such that $a_n > P$ whenever $n > N$.

Theorem: Let $\{a_n\}$ be a sequence, let $f(n) = a_n$, and suppose that $f(x)$ exists for every real number $x \geq 1$,

(i) If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) =$ L.
(ii) If $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then $\lim_{n\to\infty} f(n) = \infty$ (or $-\infty$).

Exercise 4. If $a_n = 1 + \frac{1}{n}$, determine whether $\{a_n\}$ converges or diverges.

Exercise 5. Determine whether the sequence converges or diverges:
(a) $\{ \frac{1}{4}n^2 - 1 \}$ (b) $\{ (-1)^{n-1} \}$.

Exercise 6. Determine whether the sequence $\{ \frac{5n}{e^{2n}} \}$ converges or diverges.

Theorem: (i) $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$
(ii) $\lim_{n\to\infty} |r^n| = \infty$ if $|r| > 1$

Exercise 7. List the first four terms of the sequence, and determine whether the sequence converges or diverges:
(a) $\{ (-\frac{2}{3})^n \}$
(b) $\{ (1.01)^n \}$

Exercise 8. Find the limit of the sequence $\{ \frac{2n^2}{5n^2-3} \}$.

Exercise 9. Determine whether the sequence converges or diverges. If it converges, find the limit.
(a) $a_n = \frac{n^3}{n^3+1}$
(b) $a_n = \frac{n^3}{n+1}$
(c) $a_n = \frac{3^{n+2}}{5^n}$
(d) $a_n = \sqrt{\frac{n+1}{9n+1}}$

Theorem: Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Exercise 10. Suppose the nth term of a sequence is

$$a_n = (-1)^{n+1}\frac{1}{n}.$$

Prove that $\lim_{n\to\infty} a_n = 0$.

Definition: A **sequence** is **monotonic** if successive terms are non-decreasing:

$$a_1 \leq a_2 \ ...... \leq a_n \leq ....;$$

or if they are non-increasing:

$$a_1 \geq a_2 \geq .... \geq a_n \geq ....$$

Example: An example of a monotonic sequence is $a_n = \frac{1}{n}$.

Definition: A **sequence** $\{ a_n \}$ is **bounded above** if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1.$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1.$$

If it is bounded above and below, then $\{ a_n \}$ is a **bounded sequence**.

Example: An example of a bounded sequence is $a_n = 10 - \frac{1}{n}$.

Exercise 11. Determine whether the sequence is increasing, decreasing, or monotonic. Is the sequence bounded?

(a) $a_n = (-2)^{n+1}$
(b) $a_n = \frac{2n-3}{3n+4}$
(c) $a_n = ne^{-n}$
(d) $a_n = n + \frac{1}{n}$

Theorem: A bounded, monotonic sequence has a limit.

Class Exercise 3. Determine whether the sequence converges or diverges. If it converges, find the limit.
(a) $a_n = e^{2n/(n+2)}$ (b) $a_n = \frac{(-1)^{n+1}n}{n+\sqrt{n}}$
(c) $a_n = \cos(2/n)$ (d) $a_n = \{ (\ln n)/(\ln 2n) \}$ (e) $a_n = (\tan^{-1} n)/n$ (f) $a_n = \ln(n+1)$ - $\ln n$
(g) $a_n = \sqrt[n]{2^{1+3n}}$ (h) $a_n = 2^{-n} \cos n\pi$
(i) $a_n = (\sin 2n)/(1+\sqrt{n})$ (j) $a_n = (\ln n)^2/n$
(k) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

Homework: 1-25 (every 4th), 31-59 (every 4th)