Section 11.10

Maclaurin series for $f(x)$: If a function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with radius of convergence $r > 0$, then $f^{(k)}(0)$ exists for every positive integer k and $a_n = f^{(n)}(0)/n!$. Thus,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

Taylor series for $f(x)$: If a function f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

with radius of convergence $r > 0$, then $f^{(k)}(c)$ exists for every positive integer k and $a_n = f^{(n)}(c)/n!$. Thus, $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$.

Exercise 1. e^x has the following power series representation:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

Verify that this is a Maclaurin series.

Definition: Let c be a real number and let f be a function that has n derivatives at c: $f'(c)$, $f''(c)$, ..., $f^{(n)}(c)$. The **nth-degree Taylor polynomial $P_n(x)$** of f at c is

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Taylor's formula with remainder: Let f have $n+1$ derivatives throughout an interval containing c. If x is any number in the interval that is different from c, then there is a number z between c and x such that

$$f(x) = P_n(x) + R_n(x), \text{ where } R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Theorem: Let f have derivatives of all orders throughout an interval containing c, and let $R_n(x)$ be the Taylor remainder of f at c. If $\lim_{n\to\infty} R_n(x) = 0$ for every x in the interval, then $f(x)$ is represented by the Taylor series for $f(x)$ at c.

Theorem: If x is any real number,

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0.$$

Exercise 2. Find the Maclaurin series for $\sin x$.

Exercise 3. Find the Maclaurin series for $\cos x$.

Class Exercise 1. Find a Maclaurin series for $f(x)$ using the definition of a Maclaurin series.
(a) $f(x) = \ln(1+x)$ (b) $f(x) = e^{-2x}$ (c) $f(x) = x \cos x$ (d) $f(x) = \cosh x$.

Exercise 4. Find the Taylor series for $\sin x$ in powers of $x - (\pi/6)$.

Class Exercise 2. Find the Taylor series for $f(x)$ centered at the given value of a.
(a) $f(x) = x - x^3$, $a =$ -2, (b) $f(x) = 1/x$, $a =$ -3, (c) $f(x) = \sin x$, $a = \pi/2$,
(d) $f(x) = \sqrt{x}$, $a = 16$.

Exercise 5. Use the first two nonzero terms of a Maclaurin series to approximate the following, and estimate the error in the approximation:
(a) $\sin(0.1)$ (b) $\sin x$ for any nonzero real number x in $[-1, 1]$.

Exercise 6. Approximate $\int_0^1 \sin(x^2)\, dx$ to four decimal places.

Class Exercise 3. Approximate $\int_0^1 \sin(x^4)\, dx$ to four decimal places.

Class Exercise 4. Approximate $\int_0^{0.5} x^2 e^{-x^2}\, dx$ with | error | < 0.001

Homework: 3, 7, 11, 17, 21, 25, 31, 35, 39, 41, 43