Section 11.2

Definition: An **infinite series** (or simply a **series**) is an expression of the form

$$a_1 + a_2 + \dots\dots\dots + a_n + \dots\dots,$$

or, in summation notation,

$$\sum_{n=1}^{\infty} a_n, \text{ or } \sum a_n.$$

Each number a_k is a **term** of the series, and a_n is the **nth term**.

Definition: (i) The **kth partial sum S_k** of the series $\sum a_n$ is

$$S_k = a_1 + a_2 + \dots\dots\dots + a_k.$$

(ii) The **sequence of partial sums** of the series $\sum a_n$ is

$$S_1,\ S_2,\ S_3, \dots\dots,\ S_n, \dots\dots$$

Definition A series $\sum a_n$ is **convergent** (or **converges**) if its sequence of partial sums $\{S_n\}$ converges - that is, if

$$\lim_{n\to\infty} S_n = S \text{ for some real number } S.$$

The limit S is the **sum** of the series $\sum a_n$, and we write

$$S = a_1 + a_2 + \dots\dots\dots + a_n + \dots\dots\dots$$

The series $\sum a_n$ is **divergent** (or **diverges**) if $\{\ S_n\ \}$ diverges. A divergent sequence has no sum.

Exercise 1. Given the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots\dots\dots + \frac{1}{n(n+1)} \dots\dots,$$

(a) find S_1, S_2, S_3, S_4, S_5, and S_6
(b) find S_n
(c) show that the series converges and find its sum

Class Exercise 1. Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.
(a) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$
(b) $\sum_{n=1}^{\infty} \cos \frac{1}{n^2} - \cos \frac{1}{(n+1)^2}$
(c) $\sum_{n=2}^{\infty} \frac{1}{n^3-n}$

Exercise 2. Given the series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 + (-1) + 1 + (-1) + \dots\dots\dots + (-1)^{n-1} + \dots\dots,$$

(a) find S_1, S_2, S_3, S_4, S_5, and S_6
(b) find S_n
(c) show that the series diverges.

Exercise 3. Prove that the following series is divergent:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots\dots + \frac{1}{n} + \dots\dots\dots$$

Theorem: Let $a \neq 0$. The geometric series

$$a + ar + ar^2 + \dots\dots\dots + ar^{n-1} + \dots\dots\dots$$

(i) converges and has the sum $S = \frac{a}{1-r}$ if $|r| < 1$
(ii) diverges if $|r| \geq 1$

Exercise 4. Prove that the following series converges, and find its sum:

$$0.6 + 0.06 + 0.006 + 0.0006 + ... + \frac{6}{10^n} + ...$$

Exercise 5. Prove that the following series converges, and find its sum:

$$2 + \frac{2}{3} + \frac{2}{3^2} + ... + \frac{2}{3^{n-1}} + ...$$

Class Exercise 2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.
(a) $4 + 3 + \frac{9}{4} + \frac{27}{16} + ...$
(b) $2 + 0.5 + 0.125 + 0.03125 + ...$
(c) $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$
(d) $\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$
(e) $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

Theorem: If a series $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

***n*-th term test**: (i) If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

(ii) If $\lim_{n\to\infty} a_n = 0$, then further investigation is necessary to determine whether the series $\sum a_n$ is convergent or divergent.

Exercise 6. If $\lim_{n\to\infty} a_n = 0$, can the series $\sum a_n$ diverge?

Theorem: If $\sum a_n$ and $\sum b_n$ are series such that $a_j = b_j$ for every $j > k$, where k is a positive integer, then both series converge or both series diverge.

Theorem: For any positive integer k, the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... \text{ and } \sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + ...$$

either both converge or both diverge.

Exercise 7. Show that the following series converges:

$$\frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + ... + \frac{1}{(n+2)(n+3)} + ...$$

Theorem: If $\sum a_n$ and $\sum b_n$ are convergent series with sums A and B, respectively, then
(i) $\sum (a_n + b_n)$ converges and has sum $A + B$
(ii) $\sum ca_n$ converges and has sum cA for every real number c
(iii) $\sum (a_n - b_n)$ converges and has sum $A - B$

Exercise 8. Prove that the following series converges, and find its sum:

$$\sum_{n=1}^{\infty} [\frac{7}{n(n+1)} + \frac{2}{3^{n-1}}].$$

Theorem: If $\sum a_n$ is a convergent series and $\sum b_n$ is divergent, then $\sum (a_n + b_n)$ is divergent.

Exercise 9. Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} (\frac{1}{5^n} + \frac{1}{n}).$$

Class Exercise 3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.
(a) $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$
(b) $\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$
(c) $\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n]$
(d) $\sum_{n=1}^{\infty} \frac{1}{1+(\frac{2}{3})^n}$
(e) $\sum_{k=1}^{\infty} (\cos 1)^k$
(f) $\sum_{n=1}^{\infty} (\frac{3}{5^n} + \frac{2}{n})$
(g) $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

Homework: 13, 15, 17-33 (every 4th), 39, 47, 53 - 65 (every 4th)