Section 11.3

Theorem: If $\sum a_n$ is a positive-term series and if there exists a number M such that

$$S_n = a_1 + a_2 + ... + a_n < M$$

for every n, then the series converges and has a sum $S \le M$. If no such M exists, the series diverges.

Integral Test: If $\sum a_n$ is a series, let $f(n) = a_n$ and let f be the function obtained by replacing n with x. If f is positive-valued, continuous, and decreasing for every real number $x \ge 1$, then the series $\sum a_n$

(i) converges if $\int_1^\infty f(x)\,dx$ converges
(ii) diverges if $\int_1^\infty f(x)\,dx$ diverges

Exercise 1. Use the integral test to prove that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$$

diverges.

Exercise 2. Determine whether the infinite series $\sum ne^{-n^2}$ converges or diverges.

cc **Definition**: A **p-series**, or a **hyperharmonic series**, is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ... + \frac{1}{n^p} + ...,$$

where p is a positive real number.

Theorem: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$
(i) converges if $p > 1$
(ii) diverges if $p \le 1$

Exercise 3. Determine whether the series is convergent or divergent.
(a) $\sum_{n=3}^{\infty} n^{-0.9999}$
(b) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + ...$
(c) $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + ...$
(d) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

Class Exercise 1. Determine whether the series is convergent or divergent.
(a) $\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$ (c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
(d) $\sum_{n=3}^{\infty} \frac{n^2}{e^n}$ (e) $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx.$$

Adding s_n to each side, we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \le s \le s_n + \int_n^{\infty} f(x)\,dx.$$

Exercise 4. (a) Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.
(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

Exercise 5. Use the above formula with $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Class Exercise 2. Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.

Class Exercise 3. Find the sum of the series $\sum_{n=1}^{\infty} 1/n^5$ correct to three decimal places.

Class Exercise 4. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln\ n)^2]$ would you need to add to find its sum to within 0.01?

Homework: 3, 7, 13-33 (every 4th), 39