Section 11.5

Definition: An **alternating series** is a series whose terms are alternately positive and negative.

Alternating Series Test: If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1}\, b_n = b_1 - \; b_2 + b_3 - b_4 + b_5 - b_6 + ...\; b_n > 0$$

satisfies

$$\text{(i) } b_{n+1} \leq b_n \text{ for all } n \text{ and (ii) } \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

Exercise 1. Determine whether the alternating series converges or diverges.
(a) $\sum_{n=1}^{\infty} (-1)^{n-1}\, \frac{2n}{4n^2-3}$
(b) $\sum_{n=1}^{\infty} (-1)^{n-1}\, \frac{2n}{4n-3}$

Exercise 2. Prove that the series

$$1 - \frac{1}{3!} + \frac{1}{5!} - ...... + (-1)^{n-1}\frac{1}{(2n-1)!} + ...$$

is convergent.

Exercise 3. Does the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, \frac{1}{n}$$

converge?

Class Exercise 1. Test the series for convergence or divergence.
(a) $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - ....$
(b) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - .....$
(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$
(d) $\sum_{n=1}^{\infty} (-1)^n\, \frac{n}{\sqrt{n^3+2}}$
(e) $\sum_{n=1}^{\infty} (-1)^n\, \frac{\sqrt{n}}{2n+3}$
(f) $\sum_{n=1}^{\infty} (-1)^{n+1}\, ne^{-n}$
(g) $\sum_{n=1}^{\infty} (-1)^{n-2}\, \arctan n$
(h) $\sum_{n=1}^{\infty} \frac{n\,\cos n\pi}{2^n}$
(i) $\sum_{n=1}^{\infty} (-1)^n\, \cos(\frac{\pi}{n})$
(j) $\sum_{n=1}^{\infty} (-1)^n\, (\sqrt{n+1} - \sqrt{n})$

The Alternating Series Estimation Theorem: If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1}u_n$ satisfies the three conditions of the previous theorem, then for $n \geq N$,

$$s_n = u_1 - u_2 + ... + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder $L - s_n$ has the same sign as the first unused term.

Exercise 4. Let's try the above theorem on a series whose sum we know:

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - ....$$

Exercise 5. Approximate the sum:

$$1 - \frac{1}{3!} + \frac{1}{5!} - .... + (-1)^n\frac{1}{(2n-1)!} + ...$$

to five decimal places.

Class Exercise 2. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?
(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\cdot 5^n}$ $\quad(|\text{ error }| < 0.0001)$
(b) $\sum_{n=1}^{\infty} (-1)^{n-1}\, ne^{-n}$ $\quad(|\text{ error }| < 0.01)$

Class Exercise 3. Approximate the sum of the series correct to four decimal places.
(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$
(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!}$

<u>Definition</u>: A series $\sum a_n$ is **<u>absolutely convergent</u>** if the series

$$\sum |a_n| = |a_1| + |a_2| + \dots\dots\dots + |a_n| + \dots\dots$$

is convergent.

Exercise 6. Prove that the following alternating series is absolutely convergent:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots. + (-1)^{n+1}\frac{1}{n^2} + \dots\dots$$

Exercise 7. The alternating harmonic series is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}.$$

Show that the series is (a) convergent and (b) not absolutely convergent

<u>Definition</u>: A series $\sum a_n$ is **<u>conditionally convergent</u>** if $\sum a_n$ is convergent and $\sum |a_n|$ is divergent.

<u>Theorem</u>: If a series $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

Exercise 8. Let $\sum a_n$ be the series

$$\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7} - \frac{1}{2^8} + \dots.,$$

where the signs of the terms vary in pairs as indicated and where $|a_n| = 1/2^n$. Determine whether $\sum a_n$ converges or diverges.

Exercise 9. Determine whether the following series is convergent or divergent:

$$\sin 1 + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \dots\dots + \frac{\sin n}{n^2} + \dots\dots.$$

Class Exercise 4. Determine whether the series is absolutely convergent, conditionally convergent, or divergent: (a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n}$ (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

Homework: 1-17 (every 4th), 25, 29, 33, 35, 39, 43, 47