Section 11.8

If we let $f(x) = 1/(1-x)$ with $|x| < 1$, then

$$f(x) = 1 + x + x^2 + \dots\dots\dots + x^n + \dots\dots\dots$$

We say that $f(x)$ is *represented* by this power series.

Definition: Let x be a variable. A **power series in x** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots\dots + a_n x^n + \dots\dots,$$

where each a_k is a real number.

Exercise 1. Find all values of x for which the following power series is absolutely convergent:

$$1 + \frac{1}{5}x + \frac{2}{5^2}x^2 + \dots\dots\dots + \frac{n}{5^n}x^n + \dots\dots$$

Exercise 2. Find all values of x for which $\sum \frac{1}{n!}x^n$ is convergent:

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots\dots + \frac{1}{n!}x^n + \dots\dots$$

Class Exercise 1. Find all values of x for which $\sum n!x^n$ is convergent.

Class Exercise 2. For what values of x do the following power series converge?
(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ (b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$

Theorem:

(i) If a power series $\sum a_n x^n$ converges for a nonzero number c, then it is absolutely convergent whenever $|x| < |c|$.

(ii) If a power series $\sum a_n x^n$ diverges for a nonzero number d, then it diverges whenever $|x| > |d|$.

Theorem: If $\sum a_n x^n$ is a power series, then exactly one of the following is true:
(i) The series converges only if $x = 0$.
(ii) The series is absolutely convergent for every x.
(iii) There is a number $r > 0$ such that the series is absolutely convergent if x is in the open interval $(-r, r)$ and divergent if $x < -r$ or $x > r$.

Definition: The number r is called the **radius of convergence** of the series.

Definition: The totality of numbers for which a power series converges is called the **interval of convergence**.

Exercise 3. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n$.

Definition: Let c be a real number and x a variable. A **power series in $x - c$** is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots,$$

where each a_k is a real number.

Exercise 4. Find the interval and radius of convergence of the series:

$$1 - \frac{1}{2}(x-3) + \frac{1}{3}(x-3)^2 + \cdots + (-1)^n \frac{1}{n+1}(x-3)^n + \cdots.$$

Class Exercise 3. Find the radius of convergence and interval of convergence of the series.
(a) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ (c) $\sum_{n=1}^{\infty} n^n x^n$ (d) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ (e) $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$
(f) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ (g) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$ (h) $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$ (i) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$
(j) $\sum_{n=2}^{\infty} \frac{b^n}{\ln n}(x-a)^n, b > 0$ (k) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2\cdot4\cdot6\cdot8\cdots(2n)}$ (l) $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$ (m) $\sum_{n=1}^{\infty} \frac{n!x^n}{1\cdot3\cdot5\cdots(2n-1)}$

Homework: 5-25 (every 4th), 31, 35, 39