Section 11.9

A power series $\sum a_n x^n$ determines a function f whose domain is the interval of convergence of the series. Specifically, for each x in this interval we let $f(x)$ equal the sum of the series; that is,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

If a function f is defined in this way, we say that $\sum a_n x^n$ is a **power series representation for $f(x)$** (or *of* $f(x)$). We also use the phrase f is **represented by the power series**.

Exercise 1. Find a function f that is represented by the power series

$$1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots.$$

Exercise 2. Find a power series representation for the function, $f(x) = \frac{1}{1+3x^2}$, and determine the interval of convergence.

Exercise 3. Re-express the following functions in a form that makes it easier to find the power series representation:
(a) $f(x) = \frac{x^2}{1+x^5}$ (b) $f(x) = \frac{x}{1-3x^7}$ (c) $f(x) = \frac{1}{10-2x^8}$ (d) $f(x) = \frac{3}{4+x}$

Class Exercise 1. Find a power series representation for the function and determine the interval of convergence.
(a) $f(x) = \frac{5}{1-4x^2}$ (b) $f(x) = \frac{1}{x+10}$ (c) $f(x) = \frac{x}{2x^2+1}$ (d) $f(x) = \frac{x^2}{a^3-x^3}$

Theorem: Suppose a power series $\sum a_n x^n$ has a radius of convergence $r > 0$, and let f be defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

for every x in the interval of convergence. If $-r < x < r$, then
(i) $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$
(ii) $\int_0^x f(t)\, dt = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \cdots + a_n \frac{x^{n+1}}{n+1} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Exercise 4. Use a power series representation for $1/(1+x)$ to obtain a power series representation for

$$\frac{1}{(1+x)^2} \text{ if } |x| < 1.$$

Exercise 5. Find a power series representation for $\ln(1+x)$ if $|x| < 1$.

Class Exercise 2. Find a power series representation for the function and determine the radius of convergence.
(a) $f(x) = x^2 \tan^{-1}(x^3)$ (b) $f(x) = (\frac{x}{2-x})^3$ (c) $f(x) = \frac{x^2+x}{(1-x)^3}$

Exercise 6. (a) Evaluate $\int [1/(1+x^7)]\, dx$ as a power series.
(b) Use part (a) to approximate $\int_0^{0.5} [1/(1+x^7)]\, dx$ correct to within 10^{-7}.

Class Exercise 3. Evaluate the indefinite integral as a power series. What is the radius of convergence?
(a) $\int \frac{t}{1+t^3}\, dt$ (b) $\int (\tan^{-1} x)/x\, dx$.

Class Exercise 4. Use a power series representation to approximate the definite integral to six decimal places.
(a) $\int_0^{0.4} \ln(1+x^4)\, dx$ (b) $\int_0^{0.3} \frac{x^2}{1+x^4}\, dx$

Homework: 3, 9, 13, 17, 19, 25, 29, 33