Section 12.4

Definition: If $\vec{a} = < a_1, a_2, a_3 >$ and $\vec{b} = < b_1, b_2, b_3 >$, then the **cross product** of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \; X \; \vec{b} = < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >.$$

Definition: A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Definition: A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

New Formula for Cross Product:

$$\vec{a} X \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Here is another way of expressing the last product:

$$\vec{a} X \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem: The vector $\vec{a} \; X \; \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Exercise 1. Find the cross product $\vec{u} \; X \; \vec{v}$ and $\vec{v} \; X \; \vec{u}$ if $\vec{u} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{v} = -4\vec{i} + 3\vec{j} + \vec{k}$. (Hass Sec 12.4 Ex 1)

Exercise 2. Find $\vec{a} \; X \; \vec{b}$ if $\vec{a} = < 2, -1, 6 >$ and $\vec{b} = < -3, 5, 1 >$. (Swok Sec 14.4 Ex 1)

Class Exercise 1. Find the cross product $\vec{a} \; X \; \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$. (#2,4,6)

(a) $\vec{a} = < 1, 1, -1 >$, $\vec{b} = < 2, 4, 6 >$

(b) $\vec{a} = \vec{j} + 7\vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}$

(c) $\vec{a} = t\,\vec{i} + \cos t\,\vec{j} + \sin t\,\vec{k}$, $\vec{b} = \vec{i} - \sin t\,\vec{j} + \cos t\,\vec{k}$

Definition: A vector is **orthogonal** to a plane if it is orthogonal to all of the in-plane vectors.

Exercise 3. Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$. (Hass Sec 12.4 Ex 2)

Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ (so $0 \le \theta \le \pi$), then
$| \vec{a} \; X \; \vec{b} | = | \vec{a} | \, | \vec{b} | \sin\theta$.

Corollary: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \; X \; \vec{b} = \vec{0}$.

Geometric Fact: The length of the cross product $\vec{a} \; X \; \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Exercise 4. Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$. (Hass Sec 12.4 Ex 3)

Exercise 5. Find the area of the triangle determined by $P(4, -3, 1)$, $Q(6, -4, 7)$, and $R(1, 2, 2)$. (Swok Sec 14.4 Ex 2)

Class Exercise 2. (a) Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.
(i) $P(0,0,-3)$, $Q(4,2,0)$, and $R(3,3,1)$ (#30)
(ii) $P(-1,3,1)$, $Q(0,5,2)$, and $R(4,3,-1)$ (#32)

Cross Products of Standard Basis Vectors

$$\vec{i}\ X\ \vec{j} = \vec{k} \qquad \vec{j}\ X\ \vec{k} = \vec{i} \qquad \vec{k}\ X\ \vec{i} = \vec{j}$$
$$\vec{j}\ X\ \vec{i} = -\vec{k} \qquad \vec{k}\ X\ \vec{j} = -\vec{i} \qquad \vec{i}\ X\ \vec{k} = -\vec{j}$$

Properties of the Cross Product: If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors are c is a scalar, then
1. $\vec{a}\ X\ \vec{b} = -\vec{b}\ X\ \vec{a}$
2. $(c\vec{a})\ X\ \vec{b} = c(\vec{a}\ X\ \vec{b}) = \vec{a}\ X\ (c\vec{b})$
3. $\vec{a}\ X(\vec{b} + \vec{c}) = \vec{a}\ X\ \vec{b} + \vec{a}\ X\ \vec{c}$
4. $(\vec{a} + \vec{b})\ X\ \vec{c} = \vec{a}\ X\ \vec{c} + \vec{b}\ X\ \vec{c}$
5. $\vec{a} \cdot (\vec{b}\ X\ \vec{c}) = (\vec{a}\ X\ \vec{b}) \cdot \vec{c}$
6. $\vec{a}\ X\ (\vec{b}\ X\ \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Definition: The product $\vec{a} \cdot (\vec{b}\ X\ \vec{c})$ is called the **scalar triple product** of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Formula: The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\ \vec{a} \cdot (\vec{b}\ X\ \vec{c})\ |.$$

We can write the scalar triple product as a determinant:

$$\vec{a} \cdot (\vec{b} X \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Exercise 6. Find the volume of the box determined by $\vec{u} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{v} = -2\vec{i} + 3\vec{k}$, and $\vec{w} = 7\vec{j} - 4\vec{k}$. (Hass Sec 12.4 Ex 6)

Class Exercise 3. Find the volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$: $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + \vec{j} + \vec{k}$ (#34)

Class Exercise 4. For the points $P(3,0,1)$, $Q(-1,2,5)$, $R(5,1,-1)$, and $S(0,4,2)$, find the volume of the parallelepiped with adjacent edges PQ, PR, and PS. (#36)

Fact: If $V = |\ \vec{a} \cdot (\vec{b}\ X\ \vec{c})\ | = 0$, then the vectors are coplanar.

Class Exercise 5. Use the scalar triple product to determine whether the points $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$, and $D(3,6,-4)$ lie in the same plane. (#38)

Homework: 3-43 (every 4th)