Section 12.5

A line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions the direction of a line is conveniently described by a vector, so we let $\vec{v}$ be a vector parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let $\vec{r_0}$ and $\vec{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\vec{a}$ is the vector with representation $\overrightarrow{P_0P}$, then the Triangle Law for vector addition gives $\vec{r} = \vec{r_0} + \vec{a}$. But, since $\vec{a}$ and $\vec{v}$ are parallel vectors, there is a scalar t such that $\vec{a} = t\vec{v}$. Thus,

$$\vec{r} = \vec{r_0} + t\vec{v},$$

which is a **vector equation** of L.

Exercise 1. Find a vector equation for the line that passes through $(1, 0, -2)$ and is parallel to the vector $<4,2,-1>$.

Definition: Two vectors are **equal** if and only if corresponding components are equal. Therefore, we have the three scalar equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = < a,\ b,\ c >$. Each value of the parameter t gives a point (x, y, z) on L.

Exercise 2. Find parametric equations for the line L through $P(5, -2, 4)$ that is parallel to $\vec{a} = < \frac{1}{2},\ 2,\ -\frac{2}{3} >$. (Swok Sec 14.5 Ex 1)

Exercise 3. Find parametric equations for the line through $P_1(3, 1, -2)$ and $P_2(-2, 7, -4)$. (Swok Sec 14.5 Ex 2)

Exercise 4. Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$. (Hass Sec 12.5 Ex 2)

Class Exercise 1. Find a vector equation and parametric equations for the line.
(a) The line through the point (6,-5, 2) and parallel to the vector $< 1,\ 3,\ -\frac{2}{3} >$ (#2)
(b) The line through the point (0, 14,-10) and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$ (#4)

Definition: The **symmetric equations** of L are

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Exercise 5. Find a symmetric form for the line through $P_1(3, 1, -2)$ and $P_2(-2, 7, -4)$. (Swok Sec 14.5 Ex 11)

Class Exercise 2. Find parametric and symmetric equations for the line.
(a) The line through the origin and the point $(4, 3, -1)$. (#6)
(b) The line through the points (1.0, 2.4, 4.6) and (2.6, 1.2, 0.3). (#8)
(c) The line through (2,1,0) and perpendicular to both $\vec{i} + \vec{j}$ and $\vec{j} + \vec{k}$ (#10)

Formula: The line segment from $\vec{r_0}$ to $\vec{r_1}$ is given by the vector equation

$$\vec{r}(t) = (1 - t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1.$$

Exercise 6. Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$. (Hass Sec 12.5 Ex 3)

Class Exercise 3. Find parametric equations for the line segment from $(10, 3, 1)$ to $(5, 6, -3)$. (#18)

Definition: **Skew lines** are lines that do not intersect and are not parallel.

Exercise 7. Show that the lines L_1 and L_2 with parametric equations:
$x = 1 + t \quad y = -2 + 3t \quad z = 4 - t$

$x = 2s \quad y = 3 + s \quad z = -3 + 4s$

are skew. (Stew Ex 10)

Class Exercise 4. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the points of intersection.
(a) L_1: $x = 5 - 12t$, $y = 3 + 9t$, $z = 1 - 3t$, L_2: $x = 3 + 8s$, $y = -6s$, $z = 7 + 2s$ (#20)
(b) L_1: $\frac{x}{1} = \frac{y-1}{-1} = \frac{z-2}{3}$, L_2: $\frac{x-2}{2} = \frac{y-3}{-2} = \frac{z}{7}$ (#22)

Definition: A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\overrightarrow{n}$ that is orthogonal to the plane. This orthogonal vector $\overrightarrow{n}$ is called a **normal vector**.

Definition: A **vector equation of the plane** is

$$\overrightarrow{n} \cdot (\overrightarrow{r} - \overrightarrow{r_0}) = 0.$$

Definition: A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\overrightarrow{n} = < a, b, c >$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Exercise 8. Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\overrightarrow{n} = 5\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$. (Hass Sec 12.5 Ex 6)

Exercise 9. Find a equation of the plane through the point $(5, -2, 4)$ with normal vector $\overrightarrow{a} = <1, 2, 3>$. (Swok Sec 14.5 Ex 5)

Exercise 10. Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$. (Hass Sec 12.5 Ex 7)

Class Exercise 5. Find an equation of the plane. (#24-36 even)
(a) The plane through the point (5,3,5) and with the normal vector $2\overrightarrow{i} + \overrightarrow{j} - \overrightarrow{k}$
(b) The plane through the point (2,0,1) and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$
(c) The plane through the point (2,4,6) and parallel to the plane $z = x + y$
(d) The plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$
(e) The plane through the origin and the points (2,-4,6) and (5,1,3)
(f) The plane that passes through the point (1,2,3) and contains the line $x = 3t$, $y = 1 + t$, $z = 2 - t$
(g) The plane that passes through the point (1,-1,1) and contains the line with symmetric equations $x = 2y = 3z$

Fact: If a, b, and c are not all 0, then the linear equation $ax + by + cz + d = 0$ represents a plane with normal vector $< a, b, c >$.

Definition: Two planes are **parallel** if their normal vectors are parallel.

Exercise 11. Prove that the planes $2x - 3y - z - 5 = 0$ and $-6x + 9y + 3z + 2 = 0$ are parallel. (Swok Sec 14.5 Ex 8)

Recall that if θ is the angle between two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, then

$$\cos\theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}.$$

Definition: If two planes are not parallel, then they intersect in a straight line and **the angle between the two planes** is defined as the acute angle between their normal vectors.

Exercise 12. Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$. (Hass Sec 12.5 Ex 12)

Class Exercise 6. Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.
(a) $2z = 4y - x$, $3x - 12y + 6z = 1$ (#52)
(b) $2x - 3y + 4z = 5$, $x + 6y + 4z = 3$ (#54)
(c) $x + 2y + 2z = 1$, $2x - y + 2z = 1$ (#56)

Distance Formula: The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$:

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

Exercise 13. Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$. (Hass Sec 12.5 Ex 11)

Class Exercise 7. Find the distance between the point $(-6, 3, 5)$ and the plane $x - 2y - 4z = 8$. (#72)

Class Exercise 8. Find the distance between the given parallel planes: $6z = 4y - 2x$ and $9z = 1 - 3x + 6y$. (#74)

Homework: 3-71 (every 4th), 73