Section 13.3

Theorem: Suppose that a curve has parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$. If the curve is traversed exactly once as t increases from a to b, the length of the curve is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt.$$

Exercise 1. Find the length of one arch of the cycloid that has the parametrization

$$x = t - \sin t,\ y = 1 - \cos t;\ t \in \mathbb{R}.$$

(Swok Sec 13.2 Ex 4)

Theorem: Suppose that a curve has parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$, $a \leq t \leq b$. If the curve is traversed exactly once as t increases from a to b, the length of the curve is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt.$$

Compact Form of Arc Length Formula: The arc length formula can be put into the more compact form:

$$L = \int_a^b \mid \vec{r}\,'(t) \mid dt$$

Exercise 2. Find the length of the curve. (#2)
$\vec{r}(t) = < 2t, t^2, \frac{1}{3}t^3 >$, $0 \leq t \leq 1$

Class Exercise 1. Find the length of the curve. (#4, 6)
(a) $\vec{r}(t) = \cos t\ \vec{i} + \sin t\ \vec{j} + \ln \cos t\ \vec{k}$, $0 \leq t \leq \pi/4$
(b) $\vec{r}(t) = 12t\ \vec{i} + 8t^{3/2}\vec{j} + 3t^2\vec{k}$, $0 \leq t \leq 1$

Definition: The **principal unit normal vector $\vec{N}(t)$** is $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$.

Exercise 3. Find the unit tangent and unit normal vector $\vec{T}(t)$ and $\vec{N}(t)$ for the circular motion

$$\vec{r}(t) = (\cos 2t)\vec{i} + (\sin 2t)\vec{j}.$$

(Hass Sec 13.4 Ex 3)

Class Exercise 2. Find the unit tangent and unit normal vector $\vec{T}(t)$ and $\vec{N}(t)$. (#18,20)
(a) $\vec{r}(t) = < t^2, \sin t - t\cos t, \cos t + t\sin t >$, $t > 0$ (b) $\vec{r}(t) = < t, \frac{1}{2}t^2, t^2 >$

Definition: A parametrization $\vec{r}(t)$ is called **smooth** on an interval I if $\vec{r}\,'$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$ on I. A curve is called **smooth** if it has a smooth parametrization.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in direction contribute to the rate of change of $\vec{T}$.
Definition: The **curvature** of a curve is $\kappa = \mid \frac{d\vec{T}}{ds} \mid$, where $\vec{T}$ is the unit tangent vector.

Formula: $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$.

Theorem: The curvature of the curve given by the vector function $\vec{r}$ is $\kappa(t) = \frac{|\vec{r}\,'(t) X \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$.

Exercise 4. Find the curvature of the twisted cubic $\vec{r}(t) = < t, t^2, t^3 >$ at a general point and at (0,0,0). (Stew Sec 13.3 Ex 4)

Class Exercise 3. Find the curvature: $\vec{r}(t) = t\vec{i} + t^2\vec{j} + e^t\vec{k}$. (#22)

Class Exercise 4. Find the curvature of $\vec{r}(t) = < t^2, \ln t, t\ln t >$ at the point (1,0,0). (#24)

Formula: The curvature for a curve $y = f(x)$ is $\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$.

Exercise 5. Sketch the graph of $y = 1 - x^2$, and find the curvature at the points (x, y), $(0, 1)$, $(1, 0)$, and $(2, -3)$. (Swok Sec 15.4 Ex 6)

Class Exercise 5. Find the curvature for the curve $y = \tan x$. (#28)

Definition: The vector $\overrightarrow{B}(t) = \overrightarrow{T}(t)\ X\ \overrightarrow{N}(t)$ is called the **binormal vector**.

Exercise 6. Find the unit normal and binormal vectors for the circular helix:
$\overrightarrow{r}(t) = \cos t\, \overrightarrow{i} + \sin t\, \overrightarrow{j} + t\, \overrightarrow{k}$. (Stew Sec 13.3 Ex 6)

Class Exercise 6. Find the vectors $\overrightarrow{T}$, $\overrightarrow{N}$, $\overrightarrow{B}$ for $\overrightarrow{r}(t) = <\cos t, \sin t, \ln \cos t>$ at the point (1,0,0). (#48)

Homework: 3 -19 (every 4th), 25, 29-37 ODD, 47, 49, 51