Section 14.3

Let's review derivatives from single-variable calculus.

Definition: The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided the limit exists.

Exercise 1. Find the derivative of $f(x) = x^2 + 3x$.

Class Exercise 1. Find the derivative of (a) $g(x) = x^3 + 2x^2 - 7x + 5$
(b) $h(x) = x^4 - 2x^2 + 8x - 9$

Definition: Let $g(x) = f(x, b)$. If g has a derivative at a, then we call it the

partial derivative of f with respect to x at (a, b).

Notation: The partial derivative of f with respect to x is denoted by $f_x(a, b)$.

Definition: The **partial derivative of f with respect to x at (a, b)** is

$$f_x(a, b) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Definition: The **partial derivative of f with respect to y at (a, b)** is

$$f_y(a, b) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Exercise 2. If $f(x, y) = x^3y^2 - 2x^2y + 3x$, find
(a) $f_x(x, y)$ and $f_y(x, y)$ (b) $f_x(2, -1)$ and $f_y(2, -1)$. (Swok Sec 16.3 Ex 1)

Exercise 3. Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y+\cos x}.$$ (Hass Sec 14.3 Ex 3)

Exercise 4. Find $\partial w/\partial y$ if $w = xy^2e^{xy}$. (Swok Sec 16.3 Ex 2)

Class Exercise 2. Find the partial derivatives of the function. (#16-38 even)
(a) $f(x, y) = x^4y^3 + 8x^2y$ (b) $f(x, t) = \sqrt{x}\ln t$ (c) $z = \tan xy$
(d) $f(x, y) = \frac{x}{(x+y)^2}$ (e) $w = \frac{e^v}{u+v^2}$ (f) $u(r, \theta) = \sin(r\cos\theta)$
(g) $f(x, y) = x^y$ (h) $F(\alpha, \beta) = \int_\alpha^\beta \sqrt{t^3+1}\, dt$ (i) $f(x, y, z) = x\sin(y - z)$
(j) $w = ze^{xyz}$ (k) $u = x^{y/z}$ (l) $\phi(x, y, z, t) = \frac{\alpha x + \beta y^2}{\gamma z + \delta t^2}$

Notations for Partial Derivatives: If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1f = D_xf$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2f = D_yf$$

Exercise 5. Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$ (Hass Sec 14.3 Ex 1)

Class Exercise 3. Find the indicated partial derivative. (#42,44)
(a) $f(x, y) = \arctan(y/x)$; $f_x(2, 3)$ (b) $f(x, y, z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$ $f_z(0, 0, \pi/4)$

Exercise 6. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$. (#48)
$x^2 - y^2 + z^2 - 2z = 4$.

Class Exercise 4. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$. (#50)
$yz + x \ln y = z^2$

Definition: The **second partial derivatives** of f are $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$.

Notation: $(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$

$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$

$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$
$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$

Exercise 7. Find the four second partial derivatives of $f(x, y) = 3x^4y - 2xy + 5xy^3$. (Briggs Sec 12.4 Ex 4)

Class Exercise 5. Find all the second partial derivatives. (#54, 56, 58)
(a) $f(x, y) = \sin^2(mx + ny)$ (b) $v = \frac{xy}{x-y}$ (c) $v = e^{xe^y}$

Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Exercise 8. Verify that the conclusion of Clairaut's Theorem holds.

$$f(x, y) = x^3y^2 - 2x^2y + 3x. \text{ (Swok Sec 16.3 Ex 4)}$$

Class Exercise 6. Verify that the conclusion of Clairaut's Theorem holds, that is, $u_{xy} = u_{yx}$. (#60, 62)
(a) $u = e^{xy} \sin y$ (b) $u = \ln(x + 2y)$

Exercise 9. Find f_x, f_y, and f_z when

$$f(x, y, z) = e^{-xy} \cos z. \text{ (Briggs Sec 12.4 Ex 5)}$$

Class Exercise 7. Find the indicated partial derivatives. (#64-70 even)
(a) $f(x, y) = \sin(2x + 5y)$; f_{yxy} (b) $g(r, s, t) = e^r \sin(st)$; g_{rst}
(c) $z = u\sqrt{v - w}$; $\frac{\partial^3 z}{\partial u \partial v \partial w}$ (d) $u = x^a y^b z^c$; $\frac{\partial^6 u}{\partial x \partial y^2 \partial z^3}$

Homework: 5, 7, 9, 13, 19-39 (every 4th), 41-61 (every 4th), 67