Section 14.4

Let's start the section by reviewing linearization.
Definition: If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Exercise 1. Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$.

Class Exercise 1. Find the linearization $L(x)$ of $f(x)$ at $x = a$.
(a) $f(x) = x^3 - 2x + 3$, $a = 2$ (b) $f(x) = \sqrt{x^2 + 9}$, $a = -4$

Definition: Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. The the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

Formula: Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Exercise 2. Find an equation of the tangent plane to the given surface at the specified point. (#2)
(a) $z = 3(x - 1)^2 + 2(y + 3)^2 + 7$, $(2, -2, 12)$

Class Exercise 2. Find an equation of the tangent plane to the given surface at the specified point. (#4,6)
(a) $z = xe^{xy}$, $(2, 0, 2)$
(b) $z = \ln(x - 2y)$, $(3, 1, 0)$

Definition: The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linearization of f at (a, b).

Definition: The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent line approximation** of f at (a, b).

Theorem: If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Exercise 3. Find the linearization of

$$f(x, y) = x^2 - xy + \tfrac{1}{2}y^2 + 3$$

at the point (3,2). (Hass Sec 14.6 Ex 5)

Class Exercise 3. Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point. (#12,14,16)
(a) $f(x, y) = x^3y^4$, (1,1)
(b) $f(x, y) = \sqrt{x + e^{4y}}$, (3,0)
(c) $f(x, y) = y + \sin(x/y)$, (0,3)

Definition: Let's review differentials from single-variable calculus. If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$dy = f'(x)\ dx$$

Exercise 4. (a) Find the differential dy and (b) evaluate dy for the given functions of x and dx.
$y = e^{x/10}$, $x = 0$, $dx = 0.1$.

Class Exercise 4. (a) Find the differential dy and (b) evaluate dy for the given function of x and dx.
$y = \cos \pi x$, $x = \frac{1}{3}$, $dx = -0.02$.

Definition: For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx **and** dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(x, y)\ dx + f_y(x, y)\ dy = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy.$$

Exercise 5. If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz. (Stew Sec 14.4 Ex 4)

Exercise 6. Find the differential of the function $u = \sqrt{x^2 + 3y^2}$ (#26)

Definition: Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. For such functions, the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(x - c)$$

and the linearization $L(x, y, z)$ is the right side of the equation.

Definition: The **differential** dw is defined in terms of the differentials dx, dy, dz of the independent variable by

$$dw = \frac{\partial w}{\partial x}\ dx + \frac{\partial w}{\partial y}\ dy + \frac{\partial w}{\partial z}\ dz.$$

Class Exercise 5. Find the differential of the function. (#28,30)
(a) $T = \frac{v}{1+uvw}$
(b) $L = xze^{-y^2-z^2}$

Exercise 7. A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10$ cm, a height of $h = 50$ cm, and a thickness of $t = 0.1$ cm. The manufacturing process produces tubes with a maximum error of $\pm$ 0.05 cm in the radius and height and a maximum error of $\pm$ 0.0005 cm in the thickness. The volume of the material used to construct a cylindrical tube is $V(r, h, t) = \pi ht(2r - t)$. Use differentials to estimate the maximum error in the volume of a tube. (Briggs Sec 12.7 Ex 6)

Class Exercise 6. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick. (#34)

Class Exercise 7. Four positive numbers, each less than 50, are rounded to the first decimal place and then multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from the rounding. (#40)

Homework: 3, 9, 17, 21, 25, 29, 33-43 ODD