Section 14.5

We start by reviewing the Chain Rule from single-variable calculus.

Chain Rule: If $y = f(u)$, $u = g(x)$, and the derivatives dy/du and du/dx both exist, then the composite function defined by $y = f(g(x))$ has a derivative given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u)g'(x) = f'(g(x))g'(x).$$

Exercise 1. Find $f'(x)$ if $f(x) = (7x + 3)^4$.

Class Exercise 1. Find dy/dx.
(a) $y = \sin(7 - 5x)$
(b) $y = \tan(2x - x^2)$

We now give the Chain Rule for more than one variable.

Chain Rule (Case 1): Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Exercise 2. Use a chain rule to find dw/dt if

$$w = x^2 + yz, \text{ with } x = 3t^2 + 1,\ y = 2t - 4,\ z = t^3.$$

(Swok Sec 16.5 Ex 3)

Class Exercise 2. Use the Chain Rule to find dz/dt or dw/dt. (#2,4,6)
(a) $z = \cos(x + 4y)$, $x = 5t^4$, $y = 1/t$ (b) $z = \tan^{-1}(y/x)$, $x = e^t$, $y = 1 - e^{-t}$
(c) $w = \ln(\sqrt{x^2 + y^2 + z^2})$, $x = \sin t$, $y = \cos t$, $z = \tan t$

Chain Rule (Case 2): Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Exercise 3. Use a chain rule to find $\partial w/\partial p$ and $\partial w/\partial q$ if

$$w = r^3 + s^2, \text{ with } r = pq^2,\ s = p^2 \sin q.$$

(Swok Sec 16.5 Ex 1)

Class Exercise 3. Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$. (#8, 10, 12)
(a) $z = \arcsin(x - y)$, $x = s^2 + t^2$, $y = 1 - 2st$ (b) $z = e^{x+2y}$, $x = s/t$, $y = t/s$
(c) $z = \tan\,(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

Chain Rule (General Version): Suppose that u is a differentiable function of the n variables x_1, x_2, ..., x_n and each x_j is a differentiable function of the m variables t_1, t_2,......, t_m. Then u is a function of t_1, t_2, ..., t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, ..., m$.

Exercise 4. Use a chain rule to find $\partial w/\partial z$ if

$$w = r^2 + sv + t^3, \text{ with } r = x^2 + y^2 + z^2,\ s = xyz,\ v = xe^y,\ t = yz^2.$$

(Swok Sec 16.3 Ex 2)

Class Exercise 4. Use the Chain Rule to find the indicated partial derivatives. (#22,24, 26)
(a) $T = \frac{v}{2u+v}$, $u = pq\sqrt{r}$, $v = p\sqrt{q}r$; $\frac{\partial T}{\partial p}$, $\frac{\partial T}{\partial q}$, $\frac{\partial T}{\partial r}$, when $p = 2$, $q = 1$, $r = 4$
(b) $p = \sqrt{u^2 + v^2 + w^2}$, $u = ye^x$, $v = xe^y$, $w = e^{xy}$; $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, when $x = 0$ and $y = 2$
(c) $u = xe^{ty}$, $x = \alpha^2\beta$, $y = \beta^2\gamma$, $t = \gamma^2\alpha$; $\frac{\partial u}{\partial \alpha}$, $\frac{\partial u}{\partial \beta}$, $\frac{\partial u}{\partial \gamma}$, when $\alpha = -1$, $\beta = 2$, $\gamma = 1$

Implicit Function Theorem I: We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x.

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x}{F_y}.$$

Exercise 5. Find dy/dx if $y = f(x)$ is determined implicitly by

$$y^4 + 3y - 4x^3 - 5x - 1 = 0. \text{ (Swok 16.5 Ex 5)}$$

Class Exercise 5. Find dy/dx. (#28, 30)
(a) $\cos(xy) = 1 + \sin y$ (b) $e^y \sin x = x + xy$

Implicit Function Theorem II: We suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \qquad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

Exercise 6. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = f(x, y)$ is determined implicitly by

$$x^2z^2 + xy^2 - z^3 + 4yz - 5 = 0. \text{ (Swok Sec 16.5 Ex 6)}$$

Class Exercise 6. Find $\partial z/\partial x$ and $\partial z/\partial y$. (#32, 34)
(a) $x^2 - y^2 + z^2 - 2z = 4$ (b) $yz + x \ln y = z^2$

Homework: 3, 7, 13-53 (every 4th)