Section 14.6

<u>Definition</u>: The **<u>directional derivative</u>** of f at (x_0, y_0) in the direction of a unit vector $\overrightarrow{u} = < a, b >$ is

$$D_{\overrightarrow{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

if this limit exists.

<u>Theorem</u>: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\overrightarrow{u} = < a, b >$ and

$$D_{\overrightarrow{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Exercise 1. Find the directional derivative $D_{\overrightarrow{u}} f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\overrightarrow{u}$ is the unit vector given by $\theta = \pi/6$. What is $\mathrm{D}_{\overrightarrow{u}} f(1, 2)$? (Stew Sec 14.6 Ex 2)

Class Exercise 1. Find the directional derivative of f at the given point in the direction indicated by the angle θ. (#4,6)
(a) $f(x, y) = x^3y^4 + x^4y^3$, (1,1), $\theta = \pi/6$ (b) $f(x, y) = e^x \cos y$, (0,0), $\theta = \pi/4$

<u>Definition</u> If f is a function of two variables x and y, then the **<u>gradient</u>** of f is the vector function ∇f defined by

$$\nabla f(x, y) = < f_x(x, y),\ f_y(x, y) > = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j}.$$

<u>Formula</u>: The directional derivative of a differentiable function f is

$$D_{\overrightarrow{u}} f(x, y) = \nabla f(x, y) \cdot \overrightarrow{u}.$$

Exercise 2. Let $f(x, y) = x^2 - 4xy$. (Swok Sec 16.6 Ex 2)
(a) Find the gradient of f at the point $P(1, 2)$.
(b) Use the gradient to the find the directional derivative of f at $P(1, 2)$ in the direction from $P(1, 2)$ to $Q(2, 5)$.

Class Exercise 2. (i) Find the gradient of f.
(ii) Evaluate the gradient at the point P.
(iii) Find the rate of change of f at P in the direction of the vector $\overrightarrow{u}$. (#8,10)
(a) $f(x, y) = y^2/x$, $P(1, 2)$, $\overrightarrow{u} = \frac{1}{3}(2\overrightarrow{i} + \sqrt{5}\overrightarrow{j})$
(b) $f(x, y, z) = y^2e^{xyz}$, $P(0, 1, -1)$, $\overrightarrow{u} = < \frac{3}{13}, \frac{4}{13}, \frac{12}{13} >$

<u>Definition</u>: The **<u>directional derivative</u>** of f at (x_0, y_0, z_0) in the direction of a unit vector $\overrightarrow{u} = < a, b, c >$ is

$$D_{\overrightarrow{u}} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

if this limit exists.

<u>Definition</u>: For a function f of three variables, the **<u>gradient vector</u>**, denoted by ∇f or **<u>grad</u>** f, is

$$\nabla f(x, y, z) = < f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z) >.$$

<u>Formula</u>: The directional derivative can be rewritten as

$$D_{\overrightarrow{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \overrightarrow{u}.$$

Exercise 3. Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\overrightarrow{v} = 2\overrightarrow{i} - 3\overrightarrow{j} + 6\overrightarrow{k}$. (Hass Sec 14.5 Ex 6)

Class Exercise 3. Find the directional derivative of the function at the given point in the direction of the vector $\overrightarrow{v}$. (#12, 14, 16)
(a) $f(x, y) = \frac{x}{x^2+y^2}$, (1,2), $\overrightarrow{v} = < 3, 5 >$ (b) $g(r, s) = \tan^{-1}(rs)$, (1,2), $\overrightarrow{v} = 5\overrightarrow{i} + 10\overrightarrow{j}$
(c) $f(x, y, z) = \sqrt{xyz}$, (3,2,6), $\overrightarrow{v} = < -1, -2, 2 >$

Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\overrightarrow{u}} f(\overrightarrow{x})$ is $| \nabla f(\overrightarrow{x}) |$ and it occurs when $\overrightarrow{u}$ has the same direction as the gradient vector $\nabla f(\overrightarrow{x})$.

The minimum value of the directional derivative $D_{\overrightarrow{u}} f(\overrightarrow{x})$ is $- | \nabla f(\overrightarrow{x}) |$ and it occurs when $\overrightarrow{u}$ has the opposite direction as the gradient vector $\nabla f(\overrightarrow{x})$.

Exercise 4. Let $f(x, y) = 2 + x^2 + \frac{1}{4}y^2$. Find the direction in which $f(x, y)$ increases most rapidly at the point $P(1, 2)$ and find the maximum rate of increase of f at P. (Swok Sec 16.6 Ex 3)

Class Exercise 4. Find the maximum rate of change of f at the given point and the direction in which it occurs. (#22, 24, 26)
(a) $f(s, t) = te^{st}$, (0,2) (b) $f(x, y, z) = (x + y)/z$, $(1, 1, -1)$ (c) $f(p, q, r) = \arctan(pqr)$, (1,2,1)

Definition: The **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Equation: The equation of the tangent plane is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Definition: The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane.

Exercise 5. Find an equation for the tangent plane to the ellipsoid $\frac{3}{4}x^2 + 3y^2 + z^2 = 12$ at the point $P_0(2, 1, \sqrt{6})$ and illustrate graphically. (Swok Sec 16.7 Ex 1)

Exercise 6. Find an equation of the normal line to the ellipsoid $\frac{3}{4}x^2 + 3y^2 + z^2 = 12$ at the point $P_0(2, 1, \sqrt{6})$. (Swok Sec 16.7 Ex 2)

Exercise 7. Let P_0 be the point $(3, 4 - 2)$ on the hyperboloid $16x^2 - 9y^2 + 36z^2 = 144$. Find equations for the tangent plane and the normal line at P_0. (Swok Sec 16.7 Ex 3)

Class Exercise 5. Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point. (#42, 44, 46)
(i) $y = x^2 - z^2$, (4,7,3) (ii) $xy + yz + zx = 5$, (1,2,1) (iii) $x^4 + y^4 + z^4 = 3x^2y^2z^2$, (1,1,1)

Homework: 5, 11-59 (every 4th)