Section 14.7

Let's start by reviewing local maximums and minimums from single-variable calculus.

Definition: Let a function f be defined on a set S of real numbers, and let c be a number in S.

(i) $f(c)$ is the **maximum value** of f on S if $f(x) \leq f(c)$ for every x in S.

(ii) $f(c)$ is the **minimum value** of f on S is $f(x) \geq f(c)$ for every x in S.

Theorem: If a function f has a local extremum at a number c in an open interval, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Second Derivative Test: Suppose f'' is continuous near c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Exercise 1. If $f(x) = x^5 - 5x^3$, find the local extrema of f.

Definition: A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Definition: If the inequalities in the definition above hold for all points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

Theorem: If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Second Derivatives Test: Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, $f(a, b)$ is not a local maximum or minimum.

Definition: The point (a, b) in case (c) is called a **saddle point** of f and the graph of f crosses its tangent line at (a, b).

Exercise 2. If $f(x, y) = x^2 - 4xy + y^3 + 4y$, find the local extrema and saddle points of f. (Swok Sec 16.8 Ex 3)

Class Exercise 1. Use the Second Derivative Test to classify the critical points of $f(x, y) = x^2 + 2y^2 - 4x + 4y + 6$. (Briggs Sec 12.8 Ex 2)

Exercise 3. Use the Second Derivative Test to classify the critical points of $f(x, y) = xy(x - 2)(y + 3)$. (Briggs Sec 12.8 Ex 3)

Exercise 4. Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

(Hass Sec 14.7 Ex 3)

Exercise 5. Find the local extreme values of the function

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy.$$

(Hass Sec 14.7 Ex 4)

Class Exercise 2. Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all important aspects of the function. (#6-18 even)
(a) $f(x, y) = xy - 2x - 2y - x^2 - y^2$ (b) $f(x, y) = xe^{-2x^2-2y^2}$
(c) $f(x, y) = xy(1 - x - y)$ (d) $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$ (e) $f(x, y) = y \cos x$
(f) $f(x, y) = e^y(y^2 - x^2)$ (g) $f(x, y) = \sin x \sin y$, $-\pi < x < \pi$, $-\pi < y < \pi$.

Definition: A **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

Definition: A **bounded set** in $\mathbb{R}^3$ is one that is contained within some disk.

Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Closed Interval Method: To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:
1) Find the values of f at the critical points of f in D.
2) Find the extreme values of f on the boundary of D.
3) The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Exercise 6. If $f(x, y) = x^2 - 4xy + y^3 + 4y$, find the extrema of f on the triangular region R that has vertices $(-1, -1)$, $(7, -1)$, and $(7, 7)$. (Swok Sec 16.8 Ex 4)

Exercise 7. Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$. (Hass Sec 14.7 Ex 6)

Exercise 8. Find the absolute maximum and minimum values of $f(x, y) = xy - 8x - y^2 + 12y + 160$ over the triangular region

$$R = \{\ (x, y) : 0 \le x \le 15,\ 0 \le y \le 15 - x\ \}.$$

(Briggs Sec 12.8 Ex 6)

Class Exercise 3. Find the absolute max and min values of f on the set D. (#30-36 even)
(a) $f(x, y) = x + y - xy$, D is closed triangular region with vertices (0,0), (0,2), and (4,0)
(b) $f(x, y) = 4x + 6y - x^2 - y^2$, $D = \{(x, y) \mid 0 \le x \le 4, 0 \le y \le 5\}$
(c) $f(x, y) = xy^2$, $D = \{(x, y) \mid x \ge 0, y \ge 0, x^2 + y^2 \le 3\}$
(d) $f(x, y) = x^3 - 3x - y^3 + 12y$, D is the quadrilateral whose vertices are $(-2, 3)$, (2,3), (2,2), and $(-2, -2)$.

Exercise 9. A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume. (Hass Sec 14.7 Ex 7)

Exercise 10. A rectangular box with no top is to be constructed to have a volume V = 12 ft^3. The cost per square foot of the material to be used is \$4 for the bottom, \$3 for two of the opposite sides, and \$2 for the remaining pair of opposite sides. Find the dimensions of the box that will minimize the cost. (Swok Sec 14.6 Ex 6)

Class Exercise 4. Solve the following optimization problems.
(a) Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point (0,1,1). (#40)
(b) Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin. (#42)
(c) Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.
(d) Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area. (#46)

Homework: 1, 5, 7, 15, 19, 23, 33-39 ODD, 43-55 (every 4th)