Section 15.1

Let's start by reviewing integration from single-variable calculus.

Definition of a Definite Integral: If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals or equal width $\triangle\, x = (b-a)/n$. We let x_0 $(= a)$, x_1, x_2, ..., x_n $(= b)$ be endpoints of these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then **the definite integral of f from a to b** is

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*) \triangle\, x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable on** $[a, b]$.

Exercise 1. Evaluate the integral: (a) $\int_1^2 (4x^3 - 3x^2 + 2x)\ dx$ (b) $\int_0^3 (1 + 6w^2 - 10w^4)\ dw$

Class Exercise 1. Evaluate the integral: (a) $\int_{-1}^1 t(1-t)^2\ dt$ (b) $\int_1^2 (\frac{1}{x^2} - \frac{4}{x^3})\ dx$
(c) $\int_0^4 (3\sqrt{t} - 2e^t)\ dt$ (d) $\int_1^4 \frac{\sqrt{y}-y}{y^2}\ dy$

We now define integrals for two variables.

Definition: The **definite integral of f** over the rectangle R is

$$\iint_R f(x, y)\ dA = \lim_{m,n\to\infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \triangle\, A$$

if this limit exists.

Volume as a Double Integral: If $f(x, y) \ge 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\ dA$$

Exercise 2. Estimate the volume of the solid that lies above the square $R = [0, 2]\ X\ [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes. (Stew Sec 15.1 Ex 1)

Class Exercise 2. If $R = [0, 4]\ X\ [-1, 2]$, use a Riemann sum with $m = 2$, $n = 3$ to estimate the value of $\iint_R (1 - xy^2)\ dA$. Take the sample points to be (a) the lower right corners and (b) the upper left corners of the rectangles. (#2)

Midpoint Rule of Double Integrals

$$\iint_R f(x, y)\ dA \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \triangle\, A,$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Exercise 3. Use the Midpoint Rule with $m = n = 2$ to estimate the value of integral $\iint_R (x - 3y^2)\ dA$, where $R = \{\ (x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\ \}$. (Stew Section 15.1 Ex 3)

Class Exercise 3. (a) Estimate the volume of the solid that lies below the surface $z = 1 + x^2 + 3y$ and above the rectangle $R = [1, 2]\ X\ [0, 3]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower left corners.
(b) Use the Midpoint Rule to estimate the volume in part(a). (#4)

Definition: Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b]\ X\ [c, d]$. We use the notation $\int_c^d f(x, y)\ dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y.* (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)\ dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x,y)\ dy.$$

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x)\ dx = \int_a^b \left[\int_c^d f(x,y)\ dy \right] dx.$$

The integral on the right side of the equation is called an **iterated integral**.

Exercise 4. Evaluate (a) $\int_1^4 \int_{-1}^2 (2x + 6x^2 y)\ dy\ dx$ (b) $\int_{-1}^2 \int_1^4 (2x + 6x^2 y)\ dx\ dy$.
(Swok Sec 17.1 Ex 1, 2)

Class Exercise 4. Calculate the iterated integral. (#4-14 even)
(a) $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2)\ dy\ dx$ (b) $\int_{\pi/6}^{\pi/2} \int_{-1}^5 \cos y\ dx\ dy$ (c) $\int_1^3 \int_1^5 \frac{\ln y}{xy}\ dy\ dx$
(d) $\int_0^1 \int_0^3 e^{x+3y}\ dx\ dy$ (e) $\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\ dy\ dx$ (f) $\int_0^1 \int_0^1 \sqrt{s+t}\ ds\ dt$

Fubini's Theorem: If f is continuous on the rectangle $R = \{\ (x,y) \mid a \le x \le b,\ c \le y \le d\ \}$, then

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Special Case: Suppose that $f(x,y) = g(x) \cdot h(y)$ and $R = [a,b]\ X\ [c,d]$.

$$\iint_R f(x,y)\ dA = \iint_R g(x) \cdot h(y)\ dA = \int_a^b g(x)\ dx \int_c^d h(y)\ dy, \text{ where}$$

$R = [a,b]\ X\ [c,d]$.

Exercise 5. Calculate $\iint_R f(x,y)\ dA$ for

$$f(x,y) = 100 - 6x^2 y \text{ and } R : \{(x,y) : 0 \le x \le 2,\ -1 \le y \le 1\ \}.$$

(Hass Sec 15.1 Ex 1)

Exercise 6. Evaluate $\iint_R ye^{xy}\ dA$, where $R = \{(x,y)\text{: } 0 \le x \le 1,\ 0 \le y \le \ln 2\ \}$.
(Briggs Sec 13.1 Ex 4)

Class Exercise 5. Calculate the double integral. (#16-22 even)
(a) $\iint_R (y + xy^{-2})\ dA$, $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\ \}$
(b) $\iint_R \frac{1+x^2}{1+y^2}\ dA$, $R = \{\ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$
(c) $\iint_R \frac{x}{1+xy}\ dA$, $R = [0,1]\ X\ [0,1]$ (d) $\iint_R \frac{1}{1+x+y}\ dA$, $R = [1,3]\ X\ [1,2]$

Exercise 7. Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R : 0 \le x \le 1,\ 0 \le y \le 2$. (Hass Sec 15.1 Ex 2)

Exercise 8. Find the volume of the solid bounded by the surface $f(x,y) = 4 + 9x^2y^2$ over the rectangle $R = \{\ (x,y)\text{: } -1 \le x \le 1,\ 0 \le y \le 2\ \}$. Use both possible orders of integration.
(Briggs Sec 13.1 Ex 3)

Class Exercise 6. Sketch the solid whose volume is given by the iterated integral:

$$\int_0^1 \int_0^1 (2 - x^2 - y^2)\ dy\ dx. \text{ (\#24)}$$

Class Exercise 7. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1,1]\ X\ [1,2]$. (#26)

Class Exercise 8. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$. (#28)

Class Exercise 9. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$. (#30)

Homework: 1-33 (every 4th), 43, 47, 53