Section 15.3

Let's start by reviewing polar coordinates.

In a rectangular coordinate system, the ordered pair (a, b) denotes the point whose directed distances from the x and $y$axes are b and a, respectively. Another method for representing points is to use *polar coordinates*.

To define polar coordinates, we first fix an origin O (call the **pole**) and an **initial ray** from O. Usually the positive x-axis is chosen as the initial ray. Then the point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. So we label the point P as $P(r, \theta)$.

Exercise 1. Plot the points with the given the polar coordinates.
(a) $(-3, 5\pi/6)$ (b) $(5, \tan^{-1}(4/3))$ (c) $(-1, 7\pi)$ (d) $(2\sqrt{3}, 2\pi/3)$

Class Exercise 1. Plot the point whose polar coordinates are given.
(a) $(-\sqrt{2}, 5\pi/4)$ (b) $(1, 5\pi/2)$ (c) $(2, -7\pi/6)$

Exercise 2. Sketch the graph of the following polar equation: $r = 4 \sin \theta$.

Class Exercise 2. Sketch the graph of the following polar equation: $r = 2 + 2 \cos \theta$.

Relationship between rectangular and polar coordinates: The rectangular coordinates (x, y) and polar coordinates (r, θ) of a point P are related as follows:
(i) $x = r \cos \theta$, $y = r \sin \theta$ (ii) $r^2 = x^2 + y^2$, $\tan \theta = \frac{y}{x}$ if $x \neq 0$

Exercise 3. Find the Cartesian coordinates for the following points that are given in polar coordinates: (a) $(\sqrt{2}, \pi/4)$ (b) $(1, 0)$ (c) $(0, \pi/2)$ (d) $(-\sqrt{2}, \pi/4)$

Class Exercise 3. Find the Cartesian coordinates for the following points that are given in polar coordinates: (a) $(-3, 5\pi/6)$ (b) $(5, \tan^{-1}(4/3))$ (c) $(-1, 7\pi)$ (d) $(2\sqrt{3}, 2\pi/3)$

Exercise 4. Find an equation in x and y that has the same graph as the polar equation $r = a \sin \theta$, with $a \neq 0$. Sketch the graph.

Exercise 5. Replace the following polar equations by equivalent Cartesian equations and identify their graphs.
(a) $r \cos \theta = -4$ (b) $r^2 = 4r \cos \theta$ (c) $r = \frac{4}{2\cos\theta - \sin\theta}$

Class Exercise 4. Replace the polar equation by an equivalent Cartesian equation. Then identify or describe the graph.
(a) $r \sin \theta = 0$ (b) $r \cos \theta = 0$ (c) $r = 4 \csc \theta$ (d) $r = -3 \sec \theta$ (e) $r \cos \theta + r \sin \theta = 1$
(f) $r^2 = 1$ (g) $r^2 = 4r \sin \theta$ (h) $r = \frac{5}{\sin\theta - 2\cos\theta}$ (i) $r^2 \sin 2\theta = 2$ (j) $r = \cot \theta \csc \theta$

Theorem: If f is continuous and $f(\theta) \geq 0$ on $[\alpha, \beta]$, where $0 \leq \alpha < \beta \leq 2\pi$, then the area A of the region bounded by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is

$$A = \int_\alpha^\beta \frac{1}{2} [f(\theta)]^2 \, d\theta = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta.$$

Exercise 6. Find the area of the region bounded by the cardioid $r = 2 + 2 \cos \theta$.

Class Exercise 5. Find the area A of the region R that is inside the cardioid $r = 2 + 2 \cos \theta$ and outside the region $r = 3$.

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y) \, dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$

Formula: If f is continuous on a polar region of the form

$$D = \{\ (r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\ \}$$

then

$$\iint\limits_D f(x,y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\ r\ dr\ d\theta$$

Exercise 7. Evaluate $\iint\limits_R e^{x^2+y^2}\ dy\ dx$, where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$. (Hass Sec 15.4 Ex 3)

Class Exercise 6. Evaluate the given integral by changing to polar coordinates. (#8, 10, 12, 14)
(a) $\iint\limits_R (2x - y)\ dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$
(b) $\iint\limits_R \frac{y^2}{x^2+y^2}\ dA$, where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3
(c) $\iint\limits_D \cos\sqrt{x^2+y^2}\ dA$, where D is the disk with center the origin and radius 2
(d) $\iint\limits_D x\ dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

Exercise 8. (a) Find the area of the region R that lies outside the circle $r = a$ and inside the circle $r = 2a\sin\theta$. (Swok Sec 17.3 Ex 1)
(b) Find the area of the region R bounded by one loop of the lemniscate $r^2 = a^2\sin 2\theta$, where $a > 0$. (Swok Sec 17.3 Ex 2)

Class Exercise 7. Use a double integral to find the area of the region. (#16,18)
(a) The region enclosed by both of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$
(b) The region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$

Exercise 9. Find the volume V of the solid that is bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. (Swok Sec 17.3 Ex 5)

Class Exercise 8. Use polar coordinates to find the volume of the solid. (#20, 22, 24,26)
(a) Below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane
(b) Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$
(c) Bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant
(d) Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Exercise 10. Evaluate the integral: $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\ dy\ dx$. (Hass Sec 15.4 Ex 4)

Class Exercise 9. Evaluate the iterated integral by converting to polar coordinates. (#30,32)
(a) $\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2 y\ dx\ dy$ (b) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\ dy\ dx$

Homework: 3, 9, 13, 21, 29 - 49 (every 4th)