Section 15.4

We start by reviewing moments and centers of mass from single-variable calculus.

Consider a flat plate (called a *lamina*) in the xy-plane with uniform density ρ that occupies a region R of the plate. Suppose that R is bounded by the curve $y = f(x)$.

The moment of a region R about the y-axis is:

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho\ \bar{x}_i \quad f(\bar{x}_i) \triangle\ x = \rho \int_a^b x f(x)\ dx.$$

The moment of a region R about the x-axis is:

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho\ \tfrac{1}{2}[f(\bar{x}_i)]^2 \triangle = \rho \int_a^b \tfrac{1}{2}\ [f(x)]^2\ dx.$$

The center of mass of the plate (or the centroid of R) is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\int_a^b xf(x)\ dx}{\int_a^b f(x)\ dx} \qquad \bar{y} = \frac{\int_a^b \frac{1}{2}\ [f(x)]^2\ dx}{\int_a^b f(x) dx}.$$

Exercise 1. Find the centroid of the semicircular region bounded by the x-axis on the graph of $y = \sqrt{a^2 - x^2}$ with $a > 0$.

Class Exercise 1. Find the centroid of the region bounded by the given curves.
(a) $y = \sqrt{x}$, $y = 0$, and $x = 4$
(b) $y = \sin x$, $y = 0$, $0 \le x \le \pi$
(c) $y = 2 - x^2$ and $y = x$
(d) $y = x^3$, $x + y = 2$, and $y = 0$

Formula: Suppose a lamina occupies a region D of the xy-plane and its density at point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D. Then, the total mass m of the lamina is

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \triangle\ A = \iint_D \rho(x, y)\ dA.$$

Definition: The **moment** of the entire lamina about the **x-axis**:

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^*\ \rho(x_{ij}^*, y_{ij}^*) \triangle\ A = \iint_D y \cdot \rho(x, y)\ dA.$$

Definition: The **moment** of the entire lamina about the **y-axis**:

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^*\ \rho(x_{ij}^*, y_{ij}^*) \triangle\ A = \iint_D x \cdot \rho(x, y)\ dA.$$

Formula: The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \cdot \rho(x, y)\ dA, \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \cdot \rho(x, y)\ dA,$$

where the mass m is given by

$$m = \iint_D \rho(x, y)\ dA.$$

Exercise 2. A lamina has the shape of an isosceles right triangle with equal sides of length a. The area mass density at a point P is directly proportional to the square of the distance from P to the vertex that is opposite the hypotenuse. Find the center of mass. (Swok Sec 17.6 Ex 1)

Exercise 3. A lamina has the shape of the region R in the xy-plane bounded by the parabola $x = y^2$ and the line $x = 4$. The area mass density at the point $P(x, y)$ is directly proportional to the distance from the y-axis to P. Find the center of mass. (Skow Sec 17.6 Ex 2)

Class Exercise 2. Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ. (#4,6,8,10)
(a) $D = \{\ (x, y) \mid 0 \le x \le a,\ 0 \le y \le b\ \}$; $\rho(x, y) = 1 + x^2 + y^2$
(b) D is the triangular region enclosed by the lines $x = 0$, $y = x$, and $2x + y = 6$; $\rho(x, y) = x^2$
(c) D is bounded by $y = x^2$ and $y = x + 2$; $\rho(x, y) = kx$
(d) D is bounded by the parabolas $y = x^2$ and $x = y^2$; $\rho(x, y) = \sqrt{x}$

Definition: The **moment of inertia** of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis.

Definition: The **moment of inertia of the lamina about the x-axis** is

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \, \rho(x_{ij}^*, y_{ij}^*) \triangle A = \iint_D y^2 \, \rho(x,y) \, dA.$$

Definition: The **moment of inertia of the lamina about the y-axis** is

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \, \rho(x_{ij}^*, y_{ij}^*) \triangle A = \iint_D x^2 \, \rho(x,y) \, dA.$$

Definition: The **moment of inertia of the lamina about the origin** is

$$I_o = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} [(x_{ij}^*)^2 + (y_{ij}^*)^2] \, \rho(x_{ij}^*, y_{ij}^*) \triangle A = \iint_D (x^2 + y^2) \, \rho(x,y) \, dA.$$

Exercise 4. A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\rho(x, y) = 6x + 6y + 6$. Find the plate's moments of inertia about the coordinate axes and the origin.
(Hass Sec 15.6 Ex 4)

Class Exercise 3. Suppose a lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant and the density at any point is proportional to the square of its distance from the origin. Find the moments of inertia I_x, I_y, I_0 for the lamina. (#18)

Class Exercise 4. Suppose a lamina occupies with a triangle $(0, 0)$, $(b, 0)$, and $(0, h)$ with constant density $\rho(x, y) = \rho$. Find the moments of inertia I_x and I_y. (#22)

Class Exercise 5. Suppose a lamina occupies the region under the curve $y = \sin x$ from $x = 0$ to $x = \pi$ with constant density $\rho(x, y) = \rho$. Find the moments of inertia I_x and I_y. (#24)

Homework: 5, 7, 9, 13, 15, 17, 19, 23