Section 15.6

Definition: The **triple integral of f** over the box B is

$$\iiint_B f(x,y,z)\ dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \bigtriangleup V$$

if this limit exists.

Fubini's Theorem for Triple Integrals: If f is continuous on the rectangular box $B = [a,b]$ X $[c,d]$ X $[r,s]$, then

$$\iiint_B f(x,y,z)\ dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\ dx\ dy\ dz.$$

Exercise 1. Evaluate $\iiint_Q (xy^2 + yz^3)\ dV$ if

$$Q = \{\ (x,y,z) : -1 \le x \le 1,\ 3 \le y \le 4,\ 0 \le z \le 2\ \}.$$

(Swok Sec 17.5 Ex 1)

Class Exercise 1. Evaluate the iterated integral. (#4,6,8)
(a) $\int_0^1 \int_x^{2x} \int_0^y 2xyz\ dz\ dy\ dx$ (b) $\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\ dx\ dz\ dy$
(c) $\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y\ dy\ dz\ dx$

Definition: A solid region E is said to be of **type I** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{\ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\ \},$$

where D is the projection of E onto the xy-plane.

Evaluation Theorem I: Suppose that E is a type I region. Then,

$$\iiint_E f(x,y,z)\ dV = \iint_D [\ \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ ]\ dA.$$

If the projection D of E onto the xy-plane is a type I region, then

$$E = \{\ (x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\ \}$$

and the equation becomes

$$\iiint_E f(x,y,z)\ dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dy\ dx$$

Definition A solid region E is of **type II** if it is of the form

$$E = \{\ (x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\ \},$$

where D is the projection of E onto the yz-plane.

Evaluation Theorem II: Suppose that E is a type II region. Then,

$$\iiint_E f(x,y,z)\ dV = \iint_D [\ \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx\ ]\ dA.$$

Definition: A solid region E is of **type III** if it is of the form

$$E = \{\ (x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\ \},$$

where D is the projection of E onto the xz-plane.

Evaluation Theorem III: Suppose that E is a type III region. Then,

$$\iiint_E f(x,y,z)\ dV = \iint_D [\ \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy\ ]\ dA.$$

Exercise 2. Evaluate $\iiint\limits_E z\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. (Stew Sec 15.6 Ex 2)

Exercise 3. Evaluate $\iiint\limits_E \sqrt{x^2 + z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. (Stew Sec 15.6 Ex 3)

Class Exercise 2. Evaluate the triple integral. (#10-18 even).
(a) $\iiint\limits_E e^{z/y}\, dV$, where $E = \{\ (x, y, z)\ \} \mid 0 \leq y \leq 1,\ y \leq x \leq 1,\ 0 \leq z \leq xy\ \}$
(b) $\iiint\limits_E \sin y\, dV$, where E lies below the plane $z = x$ and above the triangular region with vertices (0,0,0), $(\pi, 0, 0)$, $(0, \pi, 0)$
(c) $\iiint\limits_E xy\, dV$, where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$
(d) $\iiint_T xyz\, dV$, where T is the tetrahedron with vertices (0,0,0), (1,1,0), (1,0,1), and (1,0,0)
(e) $\iiint\limits_E z\, dV$, where E is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, $z = 0$ in the first octant

<u>**Volume Formula**</u>: $V(E) = \iiint\limits_E dV$

Exercise 4. Find the volume V of the solid that is bounded by the cylinder $y = x^2$ and by the planes $y + z = 4$ and $z = 0$. (Swok Sec 17.5 Ex 3)

Exercise 5. Find the volume of the region Q bounded by the graphs of $z = 3x^2$, $z = 4 - x^2$, $y = 0$ and $z + y = 6$. (Swok Sec 17.5 Ex 4)

Class Exercise 3. Use a triple integral to find the volume of the given solid. (#20,22)
(a) The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$
(b) The solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$

Class Exercise 4. Express the integral $\iiint\limits_E f(x, y, z)\, dV$ as an iterated integral in six different ways, where E is the solid bounded by the given surfaces. (#30,32)
(a) $y^2 + z^2 = 9$, $x = -2$, $x = 2$
(b) $x = 2$, $y = 2$, $z = 0$, $x + y - 2z = 2$

Homework: 3, 7, 13, 17, 21, 23, 25, 29, 33, 35, 39