Section 15.9

Exercise 1. Evaluate $\int_0^{\pi} \sin^2(\theta)\cos(\theta)\, d\theta$.

Substitution Rule for Definite Integrals: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Definition: Let's consider a function T whose domain D is a region in the xy-plane and whose range E is a region in the uv-plane. We call T a **transformation of coordinates** from the xy-plane to the uv-plane.

Exercise 2. Let T be the transformation of coordinates from the xy-plane to the uv-plane determined by

$$u = x + 2y,\ v = x - 2y.$$

Sketch, in the uv-plane, the vertical lines $u = 2$, $u = 4$, $u = 6$, $u = 8$ and the horizontal lines $v = -1$, $v = 1$, $v = 3$, $v = 5$. Sketch the corresponding curves in the xy-plane.
(Swok Sec 17.9 Ex 1)

Exercise 3. Consider the transformation from polar to rectangular coordinates given by

$$T\colon\ x = g(r,\theta) = r\cos\theta \text{ and } y = h(r,\theta) = r\sin\theta.$$

Find the image under this transformation of the rectangle:

$$S = \{\ (r,\theta)\colon\ 0 \le r \le 1,\ 0 \le \theta \le \pi/2.\ \}$$

(Briggs Sec 13.7 Ex 1)

Class Exercise 1. Find the images of the set S under the given transformation. (#8,10)
(a) S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$
(b) S is the disk given by $u^2 + v^2 \le 1$; $x = au$, $y = bv$.

Class Exercise 2. A region R is the xy-plane is given. Find equations for a transformation T that maps a rectangular region S in the uv-plane onto R, where the sides are parallel to the $u-$ and $v-$ axes. (#12,14)
(a) R is the parallelogram with vertices (0,0), (4,3), (2,4), $(-2,1)$
(b) R is bounded by the hyperbolas $y = 1/x$, $y = 4/x$, and the lines $y = x$, $y = 4x$ in the first quadrant

Definition: The **Jacobian** of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}.$$

Exercise 4. Compute the Jacobian of the transformation:

$$T\colon\quad x = g(r,\theta) = r\cos\theta,\qquad y = h(r,\theta) = r\sin\theta$$

(Briggs Sec 15.7 Ex 2)

Class Exercise 3. Find the Jacobian of the transformation. (#2,4)
(a) $x = uv$, $y = u/v$ (b) $x = e^{s+t}$, $y = e^{s-t}$

Definition: The **Jacobian** of the transformation T given by
$x = g(u,v,w)\quad y = h(u,v,w)\quad z = k(u,v,w)$ is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Class Exercise 4. Find the Jacobian of the transformation. (#6)
(a) $x = v + w^2$, $y = w + u^2$, $z = u + v^2$

Change of Variables in a Double Integral: Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| du\, dv$$

Exercise 5. Evaluate the integral $\iint_R \sqrt{2x(y-2x)}\, dA$, where R is the parallelogram in the xy-plane with vertices (0,0), (0,1), (2,4), (2,5). Use the transformation T: $x = 2u$, $y = 4u + v$. (Briggs Sec 13.7 Ex 3)

Class Exercise 5. Use the given transformation to evaluate the integral. (#16,18)
(a) $\iint_R (4x+8y)\, dA$, where R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$;

$$x = \tfrac{1}{4}(u+v) \qquad y = \tfrac{1}{4}(v-3u)$$

(b) $\iint_R (x^2y - xy + y^2)\, dA$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$;

$$x = \sqrt{2}u - \sqrt{2/3}v,\ y = \sqrt{2}u + \sqrt{2/3}v$$

Change of Variables in Triple Integrals:

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}\right| du\, dv\, dw$$

Exercise 6. Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz$$

by applying the transformation

$$u = (2x-y)/2, \qquad v = y/2, \qquad w = z/3$$

and integrating over an appropriate region in the uvw-space. (Hass Sec 15.8 Ex 5)

Exercise 7. Evaluate

$$\iint_R e^{(y-x)/(y+x)}\, dx\, dy,$$

where R is the region in the xy-plane bounded by the trapezoid with vertices (0,1), (0,2), (2,0), and (1,0). (Swok Sec 17.9 Ex 4)

Class Exercise 6. Evaluate the integral by making an appropriate change of variables. (#24,26)
(a) $\iint_R (x+y)e^{x^2-y^2}\, dA$, where R is the rectangle enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$, $x + y = 3$
(b) $\iint_R \sin(9x^2 + 4y^2)\, dA$, where R is in the region in the first quadrant by the ellipse $9x^2 + 4y^2 = 1$

Homework: 3-21 ODD, 23(a), 25, 29