Section 16.2

Definition: We start with a plane curve C given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

If f is defined on a smooth curve C given by the equation above, then the **line integral of f along C** is

$$\int_C f(x,y)\ ds = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*, y_i^*) \triangle s_i$$

if this limit exists.

Line Integral Formula: $\int_C f(x,y)\ ds = \int_a^b f(x(t),y(t)) \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$

Exercise 1. Evaluate $\int_C xy^2\ ds$ if C has the parametrization

$$x = \cos t, \qquad y = \sin t; \qquad 0 \le t \le \pi/2.$$

(Swok Sec 18.2 Ex 1)

Exercise 2. Evaluate $\int_C (xy + 2z)\ ds$ on the following line segments.
(a) The line segment from $P(1,0,0)$ to $Q(0,1,1)$ (b) The line segment from $Q(0,1,1)$ to $P(1,0,0)$
(Briggs Sec 14.2 Ex 3)

Class Exercise 1. Evaluate the line integral, where C is the given curve. (#2,4)
(a) $\int_C xy\ ds$, C: $x = t^2$, $y = 2t$, $0 \le t \le 1$
(b) $\int_C x \sin y\ ds$, C is the line segment from (0,3) to (4,6)

Definition: The **line integrals of f along C with respect to x and y**:

$$\int_C f(x,y)\ dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*, y_i^*) \triangle x_i \qquad \int_C f(x,y)\ dy = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*, y_i^*) \triangle y_i$$

Formula: Suppose $x = x(t)$ and $y = y(t)$

$$\int_C f(x,y)\ dx = \int_a^b f(x(t),y(t))\ x'(t)\ dt$$
$$\int_C f(x,y)\ dy = \int_a^b f(x(t),y(t))\ y'(t)\ dt$$

Line Integral Abbreviation: $\int_C P(x,y)\ dx + \int_C Q(x,y)\ dy = \int_C P(x,y)\ dx + Q(x,y)\ dy$.

Exercise 3. Evaluate $\int_C xy\ dx + x^2\ dy$ if
(a) C consists of line segments from (2,1) to (4,1) and from (4,1) to (4,5)
(b) C is the line segment from (2,1) to (4,5). (Swok Sec 18.2 Ex 4)

Class Exercise 2. Evaluate the line integral, where C is the given curve. (#6,8)
(a) $\int_C e^x\ dx$, C is the arc of the curve $x = y^3$ from $(-1,-1)$ to $(1,1)$
(b) $\int_C x^2\ dx + y^2\ dy$, C consists of the arc of the circle $x^2 + y^2 = 4$ from (2,0) to (0,2) followed by the line segment from (0,2) to (4,3)

Definition: In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t.

Line Integrals in Space: We now suppose that C is a smooth space curve given by the parametric equations: $x = x(t) \quad y = y(t) \quad z = z(t) \quad a \le t \le b$.

Definition: We define the **line integral of f along C** in a manner similar to that for plane curves:

$$\int_C f(x,y,z)\ ds = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \triangle s_i.$$

Formula

$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t)) \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\ dt.$$

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x,y,z)\ dz = \lim_{n\to\infty} f(x_i^*, y_i^*, z_i^*) \triangle z_i = \int_a^b f(x(t),y(t),z(t))\ z'(t)\ dt.$$

<u>**Line Integral Abbreviation**</u>: We evaluate integrals of the form

$$\int_C P(x,y,z)\ dx + Q(x,y,z)\ dy + R(x,y,z)\ dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

Exercise 4. Evaluate the line integral $\int_C -y\ dx + z\ dy + 2x\ dz$, where C is the helix $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$, $0 \le t \le 2\pi$. (Hass Sec 16.2 Ex 3)

Exercise 5. Evaluate $\int_C y \sin z\ ds$, where C is circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$. (Stew Sec 16.2 Ex 5)

Exercise 6. Evaluate $\int_C yz\ dx + xz\ dy + xy\ dz$ if C is given by

$$x = t,\ y = t^2,\ z = t^3;\ 0 \le t \le 2. \text{ (Swok Sec 18.2 Ex 5)}$$

Class Exercise 3. Evaluate the line integral, where C is the given curve. (#10-16 even)
(a) $\int_C xyz^2\ ds$, C is the line segment from $(-1,5,0)$ to $(1,6,4)$
(b) $\int_C (x^2+y^2+z^2)\ ds$, C: $x = t$, $y = \cos 2t$, $z = \sin 2t$, $0 \le t \le 2\pi$
(c) $\int_C y\ dx + z\ dy + x\ dz$, C: $x = \sqrt{t}$, $y = t^2$, $z = t^3$, $1 \le t \le 4$
(d) $\int_C (y+z)\ dx + (x+z)\ dy + (x+y)\ dz$, C consists of line segments from (0,0,0) to (1,0,1) and from (1,0,1) to (0,1,2).

<u>**Definition**</u>: The <u>**work**</u> W done by the force field $\vec{F}$ is the limit of Riemann sums:

$$W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\ ds = \int_C \vec{F} \cdot \vec{T}\ ds.$$

Here is the computational formula: $W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt$.

<u>**Definition**</u>: Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. Then, the <u>**line integral of $\vec{F}$ along** C</u> is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \int_c \vec{F} \cdot \vec{T}\ ds.$$

Exercise 7. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = z\vec{i} + xy\vec{j} - y^2\vec{k}$ along the curve C given by $\vec{r}(t) = t^2\vec{i} + t\vec{j} + \sqrt{t}\vec{k}$, $0 \le t \le 1$. (Hass Sec 16.2 Ex 2)

Class Exercise 4. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is given by the vector function $\vec{r}(t)$. (#20,22)
(a) $\vec{F}(x,y,z) = (x+y)\vec{i} + (y-z)\vec{j} + z^2\vec{k}$, $\vec{r}(t) = t^2\vec{i} + t^3\vec{j} + t^2\vec{k}$, $0 \le t \le 1$
(b) $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + xy\vec{k}$, $\vec{r}(t) = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$, $0 \le t \le \pi$

Exercise 8. Find the work done by the force field $\vec{F} = (y-x^2)\vec{i} + (z-y^2)\vec{j} + (x-z^2)\vec{k}$ along the curve $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $0 \le t \le 1$, from (0,0,0) to (1,1,1). (Hass Sec 16.2 Ex 4)

Exercise 9. Find the work done by the force field $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ in moving an object along the curve C parametrized by $\vec{r}(t) = \cos(\pi t)\vec{i} + t^2\vec{j} + \sin(\pi t)\vec{k}$, $0 \le t \le 1$.
(Hass Sec 16.2 Ex 5)

Class Exercise 5. Find the work done by the force field $\vec{F}(x,y) = x^2\vec{i} + ye^x\vec{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from (1,0) to (2,1). (#40)

Class Exercise 6. The force exerted by an electric charge at the origin on a charged particle at a point (x,y,z) with position vector $\vec{r} = <x,y,z>$ is $\vec{F}(\vec{r}) = K\vec{r}/|\vec{r}|^3$ where K is a constant. Find the work done as the particle moves from a straight line from (2,0,0) to (2,1,5). (#42)

<u>**Connection Between Line Integrals of Vector Fields and Line Integrals of Scalar Fields**</u>:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\ dx + Q\ dy + R\ dz, \text{ where } \vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}.$$

Homework: 1-13 ODD, 17-27 ODD, 35