Section 16.3

Definition: A curve is called **closed** if its terminal point coincides with its initial point, that is, $\overrightarrow{r}(a) = \overrightarrow{r}(b)$.

Definition: D is **open** if for every point P in D there is a disk with center P that lies entirely in D.

Definition: D is **connected** if any two points in D can be joined by a path that lies in D.

Definition: A **simple curve** is a curve that doesn't intersect itself anywhere between its end-points.

Definition: A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D.

Class Exercise 1. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected. (#32,34)
(i) $\{ (x,y) \mid 1 < |x| < 2 \}$ (ii) $\{ (x,y) \mid (x,y) \neq (2,3) \}$

Theorem If $\overrightarrow{F}(x,y) = P(x,y)\overrightarrow{i} + Q(x,y)\overrightarrow{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem: Let $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ throughout } D.$$

Then $\overrightarrow{F}$ is conservative.

Exercise 1. Determine whether the following vector field is conservative on $\mathbb{R}^2$:
$\overrightarrow{F} = < e^x \cos y, -e^x \sin y >$. (Briggs Sec 14.3 Ex 1)

Exercise 2. If $\overrightarrow{F}(x,y) = (3+2xy)\overrightarrow{i} + (x^2 - 3y^2)\overrightarrow{j}$, find a function f such that $\overrightarrow{F} = \nabla f$. (Stew Sec 16.3 Ex 4)

Class Exercise 2. Determine whether or not $\overrightarrow{F}$ is a conservative vector field. If it is, find a function f such that $\overrightarrow{F} = \nabla f$. (#4-10 even)
(a) $\overrightarrow{F}(x,y) = e^x \sin y \, \overrightarrow{i} + e^x \cos y \, \overrightarrow{j}$
(b) $\overrightarrow{F}(x,y) = (3x^2 - 2y^2)\overrightarrow{i} + (4xy+3)\overrightarrow{j}$
(c) $\overrightarrow{F}(x,y) = (2xy + y^{-2})\overrightarrow{i} + (x^2 - 2xy^{-3})\overrightarrow{j}$, $y > 0$
(d) $\overrightarrow{F}(x,y) = (xy \cosh xy + \sinh xy)\overrightarrow{i} + (x^2 \cosh xy)\overrightarrow{j}$

Definition: In general, if $\overrightarrow{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is **independent of path** if $\int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$ for any two paths C_1 and C_2 in D that have the same initial points and the same terminal points.

Theorem: $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is independent of path in D if and only if $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for every closed path C in D.

Theorem: If $\overrightarrow{F}(x,y) = M(x,y)\overrightarrow{i} + N(x,y)\overrightarrow{j}$ is continuous on an open connected region D, then the line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is independent of path if and only if $\overrightarrow{F}$ is conservative; that is, $\overrightarrow{F}(x,y) = \nabla f(x,y)$ for some scalar function f.

Exercise 3. Show that the line integral

$$\int_C (e^{3y} - y^2 \sin x)\, dx + (3xe^{3y} + 2y \cos x)\, dy$$

is independent of path in a simply connected region. (Swok Sec 18.3 Ex 3)

Exercise 4. Let $\overrightarrow{F}(x,y,z) = y^2 \cos x\ \overrightarrow{i} + (2y \sin x + e^{2z})\overrightarrow{j} + 2ye^{2z}\ \overrightarrow{k}$. Find the potential function f for $\overrightarrow{F}$. (Swok Sec 18.3 Ex 5)

Fundamental Theorem of Line Integrals: Let C be a smooth curve given by the vector function $\overrightarrow{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a)).$$

Exercise 5. Let $\overrightarrow{F}(x,y,z) = y^2 \cos x\ \overrightarrow{i} + (2y \sin x + e^{2z})\overrightarrow{j} + 2ye^{2z}\ \overrightarrow{k}$. If $\overrightarrow{F}$ is a force field, find the work done by $\overrightarrow{F}$ along any curve C from $(0, 1, \frac{1}{2})$ to $(\pi/2, 3, 2)$. (Swok Sec 18.3 Ex 5)

Class Exercise 3. (a) Find a function f such that $\overrightarrow{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ along the given curve C. (#12-18 even)
(i) $\overrightarrow{F}(x,y) = x^2\overrightarrow{i} + y^2\overrightarrow{j}$, C is the arc of the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$
(ii) $\overrightarrow{F}(x,y) = (1+xy)e^{xy}\overrightarrow{i} + x^2e^{xy}\overrightarrow{j}$, C: $\overrightarrow{r}(t) = \cos t\ \overrightarrow{i} + 2 \sin t\ \overrightarrow{j}$, $0 \le t \le \pi/2$
(iii) $\overrightarrow{F}(x,y,z) = (y^2 z + 2xz^2)\overrightarrow{i} + 2xyz\overrightarrow{j} + (xy^2 + 2x^2 z)\overrightarrow{k}$,
C: $x = \sqrt{t}$, $y = t+1$, $z = t^2$, $0 \le t \le 1$
(iv) $\overrightarrow{F}(x,y,z) = \sin y\ \overrightarrow{i} + (x \cos y + \cos z)\overrightarrow{j} - y \sin z\ \overrightarrow{k}$,
C: $\overrightarrow{r}(t) = \sin t\ \overrightarrow{i} + t\ \overrightarrow{j} + 2t\overrightarrow{k}$, $0 \le t \le \pi/2$

Class Exercise 4. Show that the line integral is independent of path and evaluate the integral.

$$\int_C \sin y\, dx + (x \cos y - \sin y)\, dy,$$

C is any path from $(2,0)$ to $(1,\pi)$. (#20)

Class Exercise 5. Find the work done by the force field $\overrightarrow{F}$ in moving an object from P to Q. (#24)

$$\overrightarrow{F}(x,y) = e^{-y}\overrightarrow{i} - xe^{-y}\overrightarrow{j};\ P(0,1),\ Q(2,0)$$

Homework: 3-11 (every 4th), 17-29 (every 4th), 31, 35-39 ODD