Section 16.4

Definition: The **positive orientation** of a simple closed curve C refers to a single counterclockwise traversal of C.

Notation: The notation

$$\oint P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C.

Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Exercise 1. Use Green's Theorem to evaluate $\oint_C 5xy\,dx + x^3\,dy$, where C is the closed curve consisting of the graphs of $y = x^2$ and $y = 2x$ between the points (0,0) and (2,4). (Swok Sec 18.4 Ex 1)

Exercise 2. Use Green's Theorem to evaluate the line integral

$$\oint_C 2xy\,dx + (x^2 + y^2)\,dy$$

if C is the ellipse $4x^2 + 9y^2 = 36$. (Swok Sec 18.4 Ex 2)

Exercise 3. Evaluate $\oint_C (4 + e^{\sqrt{x}})\,dx + (\sin y + 3x^2)\,dy$ if C is the boundary of the region R between quarter-circles of radii a and b and segments on the $x-$ and $y-$ axes. (Swok Sec 18.4 Ex 3)

Class Exercise 1. Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem. (#2,4)
(i) $\oint xy\,dx + x^2\,dy$, where C is the rectangle with vertices (0,0), (3,0), (3,1), and (0,1).
(ii) $\oint x^2y^2\,dx + xy\,dy$, C consists of the arc of the parabola $y = x^2$ from (0,0) to (1,1) and the line segments from (1,1) to (0,1) and from (0,1) to (0,0)

Class Exercise 2. Use Green's Theorem to evaluate the line integral along the given positively oriented curve. (#6,8, 10)
(a) $\int_C \cos y\,dx + x^2 \sin y\,dy$, C is the rectangle with vertices (0,0), (5,0), (5,2), (0,2)
(b) $\int_C y^4\,dx + 2xy^3\,dy$, C is the ellipse $x^2 + 2y^2 = 2$
(c) $\int_C (1 - y^3)\,dx + (x^3 + e^{y^2})\,dy$, C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

Extension of Green's Theorem: Green's Theorem can be extended to apply to regions with holes. That boundary would then consist of two simple closed curves. Let's call one simple closed curve C_1 and the other simple closed curve C_2. We assume that these boundary curves are oriented so that the region is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.

Exercise 4. If $\overrightarrow{F}$ is defined by $\overrightarrow{F}(x, y) = (-yi + x\overrightarrow{j})/(x^2 + y^2)$, show that $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin. (Stew Sec 16.4 Ex 5)

Class Exercise 3. Use Green's Theorem to evaluate $\int \overrightarrow{F} \cdot d\overrightarrow{r}$. (#12,14)
(a) $\overrightarrow{F}(x, y) = < e^{-x} + y^2, e^{-y} + x^2 >$, C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$
(b) $\overrightarrow{F}(x, y) = < \sqrt{x^2 + 1}, \tan^{-1} x >$, C is the triangle from (0,0) to (1,1) to (0,1) to (0,0)

Definition: A region is a **simple region** if it is both type I and type II.

Formula for the area of D:

$$A = \oint x\,dy = -\oint y\,dx = \frac{1}{2}\oint x\,dy - y\,dx.$$

Exercise 5. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Stew Sec 16.4 Ex 3)

Homework: 1, 3, 5, 11, 15, 19, 23, 25