Section 16.5

Definition: If $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\overrightarrow{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \overrightarrow{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{k}.$$

Mnemonic Device: curl $\overrightarrow{F} = \nabla\ X\ \overrightarrow{F}$

Exercise 1. If $\overrightarrow{F}(x,y,z) = xy^2z^4\overrightarrow{i} + (2x^2y+z)\overrightarrow{j} + y^3z^2\overrightarrow{k}$, find $\nabla\ X\ \overrightarrow{F}$. (Swok Sec 18.1 Ex 3)

Theorem: If f is a function of three variables that has continuous second partial derivatives, then

$$\text{curl } (\nabla\ f) = \overrightarrow{0}.$$

Theorem: If $\overrightarrow{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\overrightarrow{F} = \overrightarrow{0}$, then $\overrightarrow{F}$ is a conservative vector field.

Class Exercise 1. Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\overrightarrow{F} = \nabla\ f$. (#14,16,18)
(a) $\overrightarrow{F}(x,y,z) = xyz^2\overrightarrow{i} + x^2yz^2\overrightarrow{j} + x^2y^2z\overrightarrow{k}$
(b) $\overrightarrow{F}(x,y,z) = \overrightarrow{i} + \sin z\,\overrightarrow{j} + y\cos z\,\overrightarrow{k}$
(c) $\overrightarrow{F}(x,y,z) = e^x \sin yz\ \overrightarrow{i} + ze^x \cos yz\ \overrightarrow{j} + ye^x \cos yz\ \overrightarrow{k}$

Definition: If $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of $\overrightarrow{F}$** is the function of three variables defined by

$$\text{div } \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Mnemonic Device: div $\overrightarrow{F} = \nabla \cdot \overrightarrow{F}$

Exercise 2. Compute the divergence of the following vector fields. (Briggs Sec 14.5 Ex 1)
(a) $\overrightarrow{F} = < x,\ y,\ z >$ (b) $\overrightarrow{F} = < -y,\ x-z,\ y >$ (c) $\overrightarrow{F} = < -y,\ x,\ z >$

Exercise 3. Compute the divergence of the radial vector field

$$\overrightarrow{F} = \frac{\overrightarrow{r}}{|\overrightarrow{r}|} = \frac{<x,y,z>}{\sqrt{x^2+y^2+z^2}}.$$

(Briggs Sec 14.5 Ex 2)

Exercise 4. Compute the curl of the rotational field $\overrightarrow{F} = \overrightarrow{a}\ x\ \overrightarrow{r}$, where $\overrightarrow{a} = < 1, -1, 1 >$ and $\overrightarrow{r} = < x,\ y,\ z >$. What is the direction and the magnitude of the curl? (Briggs Sec 14.5 Ex 4)

Class Exercise 2. Find (a) the curl and (b) the divergence of the vector field. (#2-8 even)
(i) $\overrightarrow{F}(x,y,z) = xy^2z^3\overrightarrow{i} + x^3yz^2\overrightarrow{j} + x^2y^3z\overrightarrow{k}$ (ii) $\overrightarrow{F}(x,y,z) = \sin yz\ \overrightarrow{i} + \sin zx\ \overrightarrow{j} + \sin xy\ \overrightarrow{k}$
(iii) $\overrightarrow{F}(x,y,z) = e^{xy}\sin z\ \overrightarrow{j} + y\tan^{-1}(x/z)\ \overrightarrow{k}$ (iv) $\overrightarrow{F}(x,y,z) = < \frac{x}{y},\ \frac{y}{z},\ \frac{z}{x} >$

Theorem: If $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \overrightarrow{F} = 0.$$

Class Exercise 3. Let f be a scalar field and $\overrightarrow{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field. (#12)
(a) curl f (b) grad f (c) div $\overrightarrow{F}$ (d) curl(grad f)
(e) grad $\overrightarrow{F}$ (f) grad(div $\overrightarrow{F}$) (g) div(grad f) (h) grad(div f)
(i) curl(curl $\overrightarrow{F}$) (j) div(div $\overrightarrow{F}$) (k) (grad f) X (div $\overrightarrow{F}$) (l) div(curl(grad f))

Homework: 1- 11 ODD, 15-23 ODD