Section 16.9

Divergence Theorem: Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\overrightarrow{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E \text{div } \overrightarrow{F}\, dV.$$

Exercise 1. Consider the radial field $\overrightarrow{F} = < x, y, z >$ and let S be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region D. Assume $\overrightarrow{n}$ is the outward unit normal vector on the sphere. Evaluate both integrals of the Divergence Theorem. (Briggs Sec 14.8 Ex 1)

Class Exercise 1. Verify that the Divergence Theorem is true for the vector field $\overrightarrow{F}$ on the region E. (#2,4)
(a) $\overrightarrow{F}(x,y,z) = x^2\,\overrightarrow{i} + xy\,\overrightarrow{j} + z\,\overrightarrow{k}$, E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane
(b) $\overrightarrow{F}(x,y,z) = < x^2, -y, z >$, E is the solid cylinder $y^2 + z^2 \leq 9$, $0 \leq x \leq 2$

Exercise 2. Let Q be the region bounded by the circular cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$. Let S denote the surface of Q. If $\overrightarrow{F}(x,y,z) = x^3\,\overrightarrow{i} + y^3\,\overrightarrow{j} + z^3\,\overrightarrow{k}$, use the Divergence Theorem to find $\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS$. (Swok Sec 18.6 Ex 1)

Exercise 3. Let Q be the region bounded by the cylinder $z = 4 - x^2$, the plane $y + z = 5$, and the $xy-$ and $xz-$ planes. Let S be the surface of Q. If $\overrightarrow{F}(x,y,z) = (x^3 + \sin z)\,\overrightarrow{i}$ $+ (x^2 y + \cos z)\,\overrightarrow{j} + e^{x^2+y^2}\,\overrightarrow{k}$, find $\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS$. (Swok Sec 18.6 Ex 2)

Class Exercise 2. Use the Divergence Theorem to calculate the surface integral $\iint_S \overrightarrow{F} \cdot d\overrightarrow{S}$; that is, calculate the flux of $\overrightarrow{F}$ across S. (#6-14 even)
(a) $\overrightarrow{F}(x,y,z) = x^2yz\,\overrightarrow{i} + xy^2z\,\overrightarrow{j} + xyz^2\,\overrightarrow{k}$, S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are positive numbers

(b) $\overrightarrow{F}(x,y,z) = (x^3 + y^3)\,\overrightarrow{i} + (y^3 + z^3)\,\overrightarrow{j} + (z^3 + x^3)\,\overrightarrow{k}$, S is the sphere with center the origin and radius 2

(c) $\overrightarrow{F}(x,y,z) = z\,\overrightarrow{i} + y\,\overrightarrow{j} + zx\,\overrightarrow{k}$, S is the surface of the tetrahedron enclosed by the coordinate plane and the plane

$$\tfrac{x}{a} + \tfrac{y}{b} + \tfrac{z}{c} = 1,$$

where a, b, and c are positive numbers.

(d) $\overrightarrow{F}(x,y,z) = x^4\,\overrightarrow{i} - x^3z^2\,\overrightarrow{j} + 4xy^2z\,\overrightarrow{k}$, S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$

(e) $\overrightarrow{F} = |\ \overrightarrow{r}\ |^2\ \overrightarrow{r}$, where $\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$, S is the sphere with radius R and center the origin.

Class Exercise 3. Let $\overrightarrow{F}(x,y,z) = z\,\tan^{-1}(y^2)\,\overrightarrow{i} + z^3\ln(x^2 + 1)\,\overrightarrow{j} + z\,\overrightarrow{k}$. Find the flux of $\overrightarrow{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward. (#18)

Homework: 1-11 ODD, 17, 19