Section 6.2

Definition: A cylinder is bounded by a plane region B_1, called the base, and a congruent region B_2 in a parallel plane. The cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2.

Formula: If the area of the base is A and the height of the cylinder is h, then the volume V of the cylinder is defined as

$$V = Ah.$$

Definition: Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*) \triangle x = \int_a^b A(x)\, dx.$$

Exercise 1. Find the volume of the indicated solid.
(a) A solid has as its base the region in the xy-plane bounded by the graphs of $x = 4$ and $x = y^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a square with a base in the xy-plane.

(b) A solid has as its base the region in the xy-plane bounded by the graph of $x^2 + y^2 = 9$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an equilateral triangle with base in the xy-plane.

Class Exercise 1. Find the volume of the indicated solid.
(a) A solid has as its base the region in the xy-plane bounded by the graph $y = \sin x$ and the x-axis on the interval $[0, \pi]$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a circle with a diameter in the xy-plane.

(b) A solid has as its base the region in the xy-plane bounded by the graph $x^2 + y^2 = 4$. Find the volume of the solid if every cross section by a plane perpendicular to the y-axis is a rectangle with a base in xy-plane, and height equal to $\frac{1}{2}$ the base.

(c) A solid has as its base the region in the xy-plane bounded by the graphs of $y = 9$ and $y = x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a semicircle with a diameter in the xy-plane.

(d) A solid has as its base the region in the xy-plane bounded by the graphs of $y = x$ and $y = x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the y-axis is a square with a diagonal in the xy-plane.

Let f be continuous on $[a, b]$, and let R be the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$. The volume V of the solid of revolution generated by revolving R about the x-axis is

$$V = \int_a^b \pi\, [f(x)]^2\, dx.$$

Exercise 2. Find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis.
(a) $y = x^2 + 1$, $x = -1$, $x = 1$, x-axis
(b) $y = x^2$, $y = 0$, $x = 2$
(c) $y = x^3$, $y = 0$, $x = 2$

Class Exercise 2. Find the volume of the solid generated by revolving the region bounded by the lines and curves about the x-axis.
(a) $y = x - x^2$, $y = 0$.
(b) $y = \sin x \cos x$, $y = 0$.
(c) $y = \sqrt{9 - x^2}$, $y = 0$.
(d) $x + 2y = 2$, $x = 0$, $y = 0$.

If we partition the y-interval $[c, d]$ and use *horizontal* rectangles of width $\triangle\, y_k$ and length $g(w_k)$, the same type of reasoning that gave us the previous definition leads to the following definition:

$$V = \int_c^d \pi\ [g(y)]^2\ dy.$$

Exercise 3. The region bounded by the y-axis and the graphs of $y = x^3$, $y = 1$, and $y = 8$ is revolved about the y-axis. Find the volume of the resulting solid.

Exercise 4. Find the volume of the solid generated by revolving the region enclosed by $x = y^{3/2}$, $x = 0$, and $y = 2$ about the y-axis.

Class Exercise 3. Find the volume of the solid generated by revolving the region enclosed by $x = \tan(\frac{\pi}{4}y)$, $x = 0$, and $y = 1$ about the y-axis.

Class Exercise 4. Find the volume of the solid generated by revolving the region enclosed by $x = \sqrt{5}y^2$, $x = 0$, $y =$ -1, and $y = 1$ about the y-axis.

Since we may revolve a region about the x-axis, the y-axis, or some other line, it is not advisable to merely memorize the formulas mentioned previously. It is better to remember the following general rule for finding the volume of a circular disk:

$$V = \pi(\text{radius})^2(\text{thickness}).$$

Here are some guidelines for finding the volume of a solid of revolution using disks.

1. Sketch the region R to be revolved, and label the boundaries. Show a typical vertical rectangle of width dx or a horizontal rectangle of width dy.

2. Sketch the solid generated by R and the disk generated by the rectangle in guideline 1.

3. Express the radius of the disk in terms of x and y, depending on whether its thickness is dx or dy.

4. Use the formula above to find a formula for the volume of the disk.

5. Apply the limit of sums operator $\int_a^b$ and $\int_c^d$ to the expression in guideline 4 and evaluate the integral.

Let us next consider the region illustrated below.

If the region is revolved around the x-axis, the cross section perpendicular to the axis of revolution is a washer, a circular region with a circular region cut from its center. The area of a washer can be found by subtracting the inner area from the outer area. Here is the formula for the volume of the solid:

$$V = \int_a^b \pi\ ([f(x)]^2 - [g(x)]^2)\ dx.$$

If the region is revolved around the y-axis, the volume of the solid is:

$$V = \pi \int_c^d ((f(y))^2 - (g(y))^2)\, dy,$$

where $f(y)$ is the outer radius and $g(y)$ is the inner radius.

Exercise 5. The region bounded by the graphs of the equations $x^2 = y - 2$ and $2y - x - 2 = 0$ and by the vertical lines $x = 0$ and $x = 1$ is revolved about the x-axis. Find the volume of the resulting solid.

Exercise 6. Find the volume of the solid generated by revolving the region described in the last example about the line $y = 3$.

Exercise 7. The region in the first quadrant bounded by the graphs of $y = \frac{1}{8}x^3$ and $y = 2x$ is revolved around the y-axis. Find the volume of the resulting solid.

Exercise 8. Find the volume of the solid generated by revolving the region bounded by the lines and curves about the given axis.
(a) $y = x + 2$, $y = x^2$, x-axis.
(b) $y = 2 - 2x^2$, $y = 1 - x^2$, x-axis.

Class Exercise 5. Find the volume of the solid generated by revolving the region bounded by the lines and curves about the given axis: $y = \frac{1}{2}x + \frac{1}{2}$, $y = \frac{1}{x}$, $x = 3$, x-axis.

Class Exercise 6. Set up an integral to find the volume of the solid generated by revolving the region bounded by the lines and curves about the given axis.
(a) $y = \frac{1}{2}x^2 + 2$, $y = 0$, $x = -1$, $x = 2$, x-axis.
(b) $x = 2y^2 - 4$, $x = -2$, y-axis.
(c) $y = \cos x$, $y = \sin x$, $x = 0$, $x = \frac{\pi}{4}$, x-axis.
(d) $x = y^{2/3}$, $x - y = 2$, $y = 8$, y-axis.

Homework: 1-49 (every 4th), 55- 67 (every 4th)