Section 7.4

Definition: The method of rewriting rational functions as a sum of simpler fractions is called the **method of partial fractions**.

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.

We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known.

Definition: A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \ldots\ldots + \frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor divides $g(x)$. Then, to this factor, assign the sum of n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \ldots\ldots + \frac{B_nx+C_n}{(x^2+px+q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all of these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Exercise 1. Use partial fractions to evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\, dx$.

Class Exercise 1. Use partial fractions to evaluate the following integrals:

(a) $\int \frac{1}{1-x^2}\, dx$.
(b) $\int \frac{y}{y^2-2y-3}\, dy$.
(c) $\int \frac{1}{t^3+t^2-2t}\, dt$.

Exercise 2. Use partial fractions to evaluate the following integrals:
(a) $\int \frac{6x+7}{(x+2)^2}\, dx$.
(b) $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\, dx$.
(c) $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx$.
(d) $\int \frac{1}{x(x^2+1)^2}\, dx$.

Class Exercise 2. Use partial fractions to evaluate the following integrals:

(a) $\int \frac{s^3}{s^2+4}\ ds.$
(b) $\int \frac{5x^2}{x^2+x+1}\ dx.$
(c) $\int \frac{1}{(x^2-1)^2}\ dx.$
(d) $\int \frac{2}{r^2-2r+2}\ dr.$
(e) $\int \frac{x^2-2x-2}{x^3-1}\ dx.$
(f) $\int \frac{3x^2-2x+12}{(x^2+4)^2}\ dx.$
(g) $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\ dx.$
(h) $\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\ dx.$
(i) $\int \frac{x^2-x-21}{2x^3-x^2+8x-4}\ dx.$

Homework: 1-17 (every 4th), 21, 27, 31, 35, 43, 47 53, 59

Section 7.6

In this section, we use a table of integrals to evaluate integrals.

Exercise 3. Evaluate the following integrals:
(a) $\int x^3 \cos x\ dx.$
(b) $\int \frac{1}{x^2\sqrt{3+5x^2}}\ dx.$
(c) $\int (\sin 2x)/(\sqrt{3-5\cos x})\ dx.$

Class Exercise 3. Evaluate the following integrals:
(a) $\int_0^2 x^2\ \sqrt{4-x^2}\ dx$
(b) $\int \ln(1+\sqrt{x})/\sqrt{x}\ dx$
(c) $\int \frac{\sqrt{2y^2-3}}{y^2}\ dy$
(d) $\int x^2\ \text{csch}(x^3+1)\ dx$
(e) $\int \sin^{-1}\ \sqrt{x}\ dx$
(f) $\int x \sin\ (x^2)\ \cos\ (3x^2)\ dx$
(g) $\int \frac{1}{2x^3-3x^2}\ dx$
(h) $\int \sin 2\theta/\sqrt{5-\sin\theta}\ d\theta$
(i) $\int_0^2 x^3\sqrt{4x^2-x^4}\ dx$
(j) $\int \sin^6 2x\ dx$
(k) $\int_0^1 x^4 e^{-x}\ dx$
(l) $\int (t+1)\sqrt{t^2-2t-1}\ dt$
(m) $\int e^t\ \sin(\alpha\ t\ \text{-}3)\ dt$

Homework: 1, 7-39 (every 4th)