Section 7.8

Definition: Integrals with infinite limits of integration are **improper integrals**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\, dx = \lim_{a\to-\infty} \int_a^b f(x)\, dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx,$$

where c is any real number.

In parts 1 and 2 , if the limit is finite the improper integral **converges** and the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**. In part 3, the integral on the left-hand side of the equation **converges** if both improper integrals on the right-hand side converge, otherwise it **diverges** and has no value.

Exercise 1. Does the improper integral $\int_1^\infty \frac{1}{x}\, dx$ converge or diverge?

Exercise 2. Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx$.

Exercise 3. For what values of p does the integral $\int_1^\infty \frac{1}{x^p}\, dx$ converge? When the integral does converge, what is its value?

Definition: Integrals of functions that become infinite at a point within the interval of integration are **improper integrals**.

1. If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)\, dx = \lim_{c\to a^+} \int_c^b f(x)\, dx.$$

2. If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)\, dx = \lim_{c\to b^-} \int_a^c f(x)\, dx.$$

3. If $f(x)$ is continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

In parts 1 and 2, if the limit is finite the improper integral **converges** and the limit is the **value** of the improper integral. If the limit fails to exist the improper integral **diverges**. In part 3, the integral on the left-hand side of the equation **converges** if both integrals on the right-hand side have values, otherwise it **diverges**

Exercise 4. Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx$.

Exercise 5. Evaluate $\int_1^4 \frac{1}{x-2}\, dx$.

Class Exercise 1. Evaluate the integral or state that it diverges.
(a) $\int_1^\infty \frac{1}{x^{1.001}}\, dx$
(b) $\int_0^4 \frac{1}{\sqrt{4-r}}\, dr$
(c) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$
(d) $\int_{-\infty}^{-2} \frac{2}{x^2-1}\, dx$
(e) $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\, d\theta$
(f) $\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\, dx$
(g) $\int_1^2 \frac{1}{s\sqrt{s^2-1}}\, ds$
(h) $\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\, dx$
(i) $\int_{-\infty}^0 \theta\, e^\theta\, d\theta$
(j) $\int_{-\infty}^\infty e^{-|x|}\, dx$

Exercise 6. Does the integral $\int_1^{\infty} e^{-x^2}\, dx$ converge?

Theorem: Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then
1. $\int_a^{\infty} f(x)\, dx$ converges if $\int_a^{\infty} g(x)\, dx$ converges

2. $\int_a^{\infty} g(x)\, dx$ diverges if $\int_a^{\infty} f(x)\, dx$ diverges

Exercise 7. Does $\int_1^{\infty} \sin^2 x/x^2\, dx$ converge or diverge?

Exercise 8. Does $\int_1^{\infty} \frac{1}{\sqrt{x^2-0.1}}\, dx$ converge or diverge?

Theorem: If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^{\infty} f(x)\, dx \text{ and } \int_a^{\infty} g(x)\, dx \text{ both converge or both diverge.}$$

Exercise 9. Show that $\int_1^{\infty} \frac{1}{1+x^2}\, dx$ converges by comparison with $\int_1^{\infty} \frac{1}{x^2}\, dx$. Find and compare the two integral values.

Exercise 10. Show that $\int_1^{\infty} \frac{1}{e^x-5}\, dx$ converges.

Class Exercise 2. Use integration, the direct comparison test, or the limit comparison test to determine whether the integral converges or diverges.
(a) $\int_0^{\pi/2} \tan\theta\, d\theta$
(b) $\int_{-\infty}^{\infty} 2xe^{-x^2}\, dx$
(c) $\int_0^{\pi} \frac{1}{\sqrt{t}+\sin t}\, dt$
(d) $\int_1^{\infty} \frac{1}{x^3+1}\, dx$
(e) $\int_0^{2} \frac{1}{1-x}\, dx$
(f) $\int_1^{\infty} \frac{1}{1+e^{\theta}}\, d\theta$
(g) $\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\, dx$
(h) $\int_{\pi}^{\infty} \frac{2+\cos x}{x}\, dx$
(i) $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\, dx$
(j) $\int_0^{\infty} \frac{1}{(1+y^2)(1+\tan^{-1} y)}\, dy$

Homework: 15, 19, 25-45 (every 4th), 57, 61. 63. 67