Section 3.2

In the last section, we learned that one common way to model a population is with a differentiable function P growing at a rate proportional to the size of the population. Thus, for some constant k,

$$\frac{dP}{dt} = kP.$$

Notice that

$$\frac{dP/dt}{P} = k$$

is constant.

Definition: The rate is called a **relative growth rate**.

Definition: Population growth is sometimes given by the differential equation

$$\frac{dP}{dt} = Pf(P).$$

The differential equation is called the **density-dependent hypothesis**.

Definition: Suppose an environment is capable of sustaining no more than a fixed number K of individuals in its population. The quantity K is called the **carrying capacity** of the environment.

Assuming that an environment has a carrying capacity K, the population growth is modeled by the differential equation

$$\frac{dP}{dt} = P(r - \frac{r}{K}P).$$

Definition: The above equation can be rewritten as follows:

$$\frac{dP}{dt} = P(a - bP).$$

This equation is known as the **logistic equation**.

Definition: The solution is called a **logistic function**.

Definition: The graph of a logistic function is called a **logistic curve**.

Formula: The solution to the logistic equation with $P(0) = P_0$ is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

Exercise 1. Taking the 1790 population of 3.93 million as the initial population and given the 1840 and 1890 population of 17.07 million and 62.98 million, respectively, use the logistic model to estimate the population at time t. (Nagle Section 3.2 Example 4)

Class Exercise 1. The number $N(t)$ of people in a community who are exposed to a particular advertisement is governed by the logistic equation. Initially, $N(0) = 500$, and it is observed that $N(1) = 1000$. Solve for $N(t)$ if it is predicted that the limiting number of people in the community who will see the advertisement is 50,000. (#2)

Definition: The differential equations

$$\frac{dP}{dt} = P(a - bP) - \text{h} \text{ and } \frac{dP}{dt} = P(a - bP) + h$$

could serve as models for the population in a fishery where fish are **harvested** or are **restocked** at rate h.

Definition: A chemical reaction governed by the nonlinear differential equation

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X),$$

is said to be a **second-order reaction**.

Exercise 2. A compound C is formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B is used. It is observed that 30 grams of the compound C is formed in 10 minutes. Determine the amount of C at time t if the rate of the reaction is proportional to the amounts of A and B remaining and if initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes? Interpret the solution as $t \to \infty$. (Zill Section 3.2 Example 2)

Class Exercise 2. Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 100 grams of A and 50 grams of B, and for each gram of B, 2 grams of A is used. It is observed that 10 grams of C is formed in 5 minutes. How much is formed in 20 minutes? What is the limiting amount of C after a long time? How much of chemicals A and B remains after a long time? At what time is chemical C half-formed? (#10)

Homework: 1, 3, 5, 9