Section 4.2

Conditional Probability

Exercise 1. Suppose you roll 2 dice. We know that the probability of rolling a ten is $3/36 = 1/12$. However, what is the probability that the sum of the dice is ten if we know that the faces on the dice are the same?
Let A = the event that the sum is ten.
Let B = the event that the faces are the same.

Let's draw a Venn Diagram of the situation:

Since we know that the faces are the same, we can cross out everything except B.

There are outcomes in event B and there is outcome contained in A and B. Since all outcomes are equally likely, the probability is

Remark: When we condition on an event, we are no longer concerned about the original sample space. Rather, we are concerned with the outcomes that are part of the event on which we are conditioning!

Notation: The probability of an event A given an event B is denoted by P(A|B). For the last exercise, P(A|B) = 1/6.

Remark: P(B|A) is not the same as P(A|B). The following exercise illustrates this point.

Exercise 2. What is the probability that the faces are the same if we know that the sum is ten?

Let's draw a Venn Diagram of the situation:

Since we know that the sum is ten, we can cross out everything except A.

There are outcomes in event A and there is outcome contained in A and B. Since all outcomes are equally likely, the probability is Using the new notation, P(B|A) =

Remark: Note that the answer to the last exercise, $\frac{1}{3}$, is not the same as P(A|B) = 1/6.

Exercise 3. Suppose there are five marbles in a box, three of which are black and two of which are red. Suppose you take two marbles out of the box. What is the probability that the second marble taken out is black if the first marble taken out is red?
Let A = the event that the first marble is red.
Let B = the event that the second marble is black.

Here is a picture of the situation:

+ = red
- = black

Suppose the first marble taken out is

Now, the probability of pulling out a black marble is

Using conditional probability notation,

Exercise 4. Suppose we take two marbles out of the same box (look at the previous example). What is the probability that the first marble is red and the second marble is black?

Using the intuition from the last exercise, we have the following formula:
Formula 1: For two events A and B, P(A and B) = P(A)·P(B|A).

Remark*: Keep in mind that P(A and B) = P(B and A). For example, suppose you toss a coin and then roll a die. The probability that the coin will land on heads and the die will land on 5 is the same as the probability the die will land on 5 and the coin will land on heads.

Exercise 5. The probability that it is sunny on a given day is 0.2 and the probability that it isn't sunny is 0.8. The probability that the Giants win on sunny days is 0.6 and the probability that the Giants win on non-sunny days is 0.7. Given this information, what is the probability that the Giants will win tomorrow? (First, establish the events and then go from there.)

Let event A =

Let event B =

Let event C =

There are two paths to a Giants victory: either the Giants win on a sunny day or the Giants win on a not sunny day. In other words,

The probability that the Giants win is

The following tree diagram illustrates the mathematics above:

There are two paths to a Giants victory.

P(Giants win) = $0.2 \cdot 0.6 + 0.8 \cdot 0.7 = 0.68$.

Formula 2: From Formula 1, P(A and B) = P(A)·P(B|A). Dividing both sides by P(A) yields

$$\mathrm{P(B|A)} = \frac{\mathrm{P(A\ and\ B)}}{\mathrm{P(A)}}.$$

Formula 2 is used in the next exercise.

Exercise 6. What is the probability that the sum of two dice is ten if the faces are the same? Use Formula 2 to do this problem.

Let A =

Let B =

Independence

Definition: Events A and B are **independent** whenever P(B|A) = P(B).

Remark: You already have some intuitive notions of what independence is. For example, the event that the 49ers win the Super Bowl is independent of the event that it will rain in London tomorrow. However, for the purposes of the class, it is also important to understand the mathematics of independence.

Exercise 7. Two dice are rolled. Are the following two events independent: A = the event that the faces are the same and B = the event that the sum is 10?

Since B = , P(B) is

From the last exercise, P(B|A) =

Since P(B) $\neq$ P(B|A), B and A are

Exercise 8. The following table gives the distribution, approximated to the nearest millions, of the population of the United States according to age and gender for the year 1995.

	Males	Females	Row Total
Under 25 years old	48	46	94
25 years old and older	79	90	169
Column Total	127	136	263

Let event M = male and event U = under 25. Are M and U independent events?

P(M) =

P(M|U) =

Since , M and U are

Remark: Combining the definition of independence and Formula 1 yields the following:

If A and B are independent, P(A and B) = P(A|B)·P(B) = P(A)·P(B).

Remark: Two events A and B are independent if and only if P(A and B) = P(A)·P(B). The following exercise will illustrate the remarks above.

Exercise 9. What is the probability of tossing two heads in a row?

Let A =

Let B =

Since A and B are independent,

P(A and B) =

Definition: A **compound event** is any event combining two or more simple events.

Notation: $P(A \text{ or } B) = P(\text{in a single trial, event } A \text{ occurs or event } B \text{ occurs or they both occur})$.

Notation: $P(\text{A and } B)$ denotes the probability that A and B occur in the same trial.

Exercise 10. Two dice are rolled. Define the following three events:
A = the sum of the faces is three,
B = the sum of the faces is a multiple of three,
C = the sum of the faces is a multiple of four.

(a) What is P(A)?

Method: The following method is sometimes used for dice problems. Write out the numbers 1, 2, 3, 4, 5, 6 horizontally and vertically and place a dot for each pair of numbers. Circle the dot corresponding to each outcome in the event.

(b) What is P(B)?

6

5

4

3

2

1

. 1 2 3 4 5 6

There are outcomes that belong to event B. Therefore, P(B) =

(c) What is P(C)?

6

5

4

3

2

1

. 1 2 3 4 5 6

There are outcomes that belong to event C. Therefore, P(C) =

<u>Definition</u>: The event B and C consists of the outcomes that belong to event B **<u>and</u>** event C.

(d) What is P(B and C)?

Which outcome(s) belongs to B and C? .

Therefore, P(B and C)

<u>Definition</u>: The event B or C consists of the outcomes that belong to the event B **<u>or</u>** the event C.

(e) Find P(B or C). Does P(B or C) = P(B) + P(C)?

There are distinct outcomes that belong to B or C.

Therefore, P(B or C) =

However, P(B) + P(C) =

So,

Addition Formula:

For two events B and C, P(B or C) = P(B) + P(C) - P(B and C).

The following **Venn Diagram** illustrates the **Addition Formula**:

Since B and C is contained in B and in C, we subtract off B and C to avoid double counting.

Notation: $B \cup C$ denotes B or C and $B \cap C$ denotes B and C.

Remark: With this new notation, the Addition Formula becomes:

$$P(B \cup C) = P(B) + P(C) - P(B \cap C).$$

Exercise 11. Two dice are rolled yet again. Define the following two events:
A = the sum of the faces is six,
B = the faces are the same.
Using the Addition Formula, find $P(A \cup B)$.

A =

B =

$A \cap B$ =

$P(A \cup B) = P(A) + P(B) - P(A \cap B) =$

Definition: Events A and B are **disjoint** (or **mutually exclusive**) if they cannot occur at the same time.

Exercise 12. Define the following events:
A = the sum of the faces is six,
B = the faces are the same.

Are A and B mutually exclusive events?

Exercise 13. (a) Define the following events:
A = the sum of the faces is six,
C = the sum of the faces is odd.
Are A and C mutually exclusive events?

Suppose A and B are mutually exclusive. This means that there are no outcomes that belong to $A \cap B$. Therefore, $P(A \cap B) = 0$.

Combining the last fact with the Inclusion-Exclusion Formula,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B).$$

The following Venn Diagram illustrates what happens when A and B are mutually exclusive.

<u>Fact</u>: P(A $\cup$ B) = P(A) + P(B) if and only if A and B are mutually exclusive.

(b) What is the probability that the sum of the faces is six OR the sum of the faces is odd? From part (a), we let A = the event that the sum of the faces is six and C = the event that the sum of the faces is odd.

We also know from part (a) that A and C are

Exercise 14. Suppose we flip a coin and roll a die at the same time. Define the following events:
A = the coin lands on heads,
B = the die lands on 3.

Are A and B mutually exclusive events?

Intuitively, it is possible for a coin to land on heads and a die to land on 3 at the same time. Therefore, the events are <u>not</u> mutually exclusive.

Class Exercise 1. Given $P(A) = 0.7$ and $P(B) = 0.4$:
(a) Can events A and B be mutually exclusive? Explain.
(b) If $P(A \text{ and } B) = 0.2$, compute $P(A \text{ or } B)$. (#4)

Class Exercise 2. Given $P(A) = 0.7$ and $P(B) = 0.8$:
(a) If A and B, are independent events, compute $P(A \text{ and } B)$. (#6)
(b) If $P(B|A) = 0.9$, compute $P(A \text{ and } B)$.

Class Exercise 3. Given $P(A^c) = 0.8$, $P(B) = 0.3$, $P(B|A) = 0.2$;
(a) Compute $P(A \text{ and } B)$. (#8)
(b) Compute $P(A \text{ or } B)$.

Exercise 15. Determine whether the events are disjoint:

Randomly selecting a full-time mathematics faculty member on the Evergreen Valley College campus.

Randomly selecting a Sri Lankan American on the Evergreen Valley College campus.

Exercise 16. Determine whether the events are disjoint:

Randomly selecting an Evergreen Valley College student.
Randomly selecting a current resident of Turkey.

Class Exercise 4. Determine whether the events are disjoint:

(a) Randomly selecting a fruit fly with red eyes
Randomly selecting a fruit fly with sepian (dark brown) eyes

(b) Randomly selecting a college graduate
Randomly selecting someone who is homeless

(c) Randomly selecting a physician at Bellevue Hospital in New York City and getting a surgeon
Randomly selecting a physician at Bellevue Hospital in New York City and getting a female.

Student Feedback

My teaching methods are (I hope) continually subject to improvement. If you have any comments, suggestions, or ideas, please email them to me at Sithparran.Vanniasegaram@evc.edu.

Homework

C problems

Section 4.2: 3-7 ODD, 15-35 ODD

B problems

Section 4.2: 1

A problems

Section 4.2: 9-13 ODD, 37-51 ODD