MATH 105 - Assignment 2, 2004

NOTE: You have the choice of either doing this assignment yourself or working with one other person. If you choose to work with another person, only one assignment should be submitted (to either of your tutors) and you will both be given the same mark. Apart from this, the assignment must be your own work. Any cases of copying will result in offenders getting 0 marks and may lead to further disciplinary action. You should not approach tutors for help with the questions.

Instructions for handing in the assignment

- This assignment must be handed in by 3.00 pm, **Monday 20 September**.
- It should be put in the assignment box for your tutorial group (according to its label).
- The assignment boxes are located on level 2 of the MSCS Building.
- Before handing in your assignment, complete the cover page (see over) and attach it to your assignment.

Instructions for carrying out the assignment

- Attempt all parts of each question; there is no choice of questions in this assignment.
- Partial credit will be given. If you do part of a question correctly but do not complete the question, you will be given credit for what you did correctly.
- Do not just write down answers. Show your working. **Where explanation is needed it must be given.** Thus you must state why certain procedures such as L'Hôpital's rule apply. However, be reasonably concise. Do not ramble on.
- Correct mathematical statements are required. A correct answer with incorrect calculations or reasoning will lose you marks.
- Marks will be deducted for poor presentation, including untidiness or illegibility.
- Write proper English, in sentences and paragraphs.
- MAPLE solutions should be edited to remove any unnecessary material and to allow for comments.

MATH 105

Assignment 2 - 2004

NAME(S):	
STUDENTS ID(S):	
TUTOR(S):	
TUTORIAL GROUP(S):	

STAPLE this page to the front of your assignment.

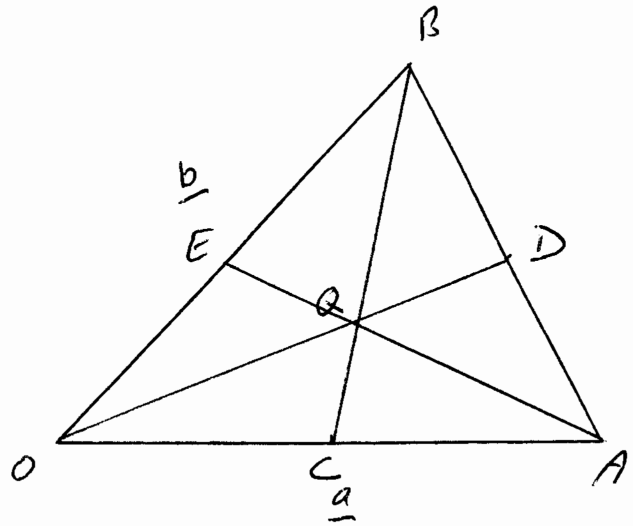
Set: **Wednesday 11 August**

Due: **3.00 pm, Monday 20 September**

Please post your assignment in the **CORRECT** box, labelled according to tutor and tutorial group.

Q1. **[6 marks]** Vector methods can be used to prove results in geometry. The following theorem (which you will be asked to prove) is typical.

Theorem: The three medians of a triangle intersect at one point.



Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Find the parametric equation of the line through B and C.

(b) Similarly, find the equation of the line through A and E.

(c) If Q is the point of intersection of these two lines, use (a) and (b) to find $\overrightarrow{OQ}$.

(d) Finally show that the point D which bisects AB lies on the line through O and Q.

Q2. **[8 marks]**

(a) Use the Principle of Induction to show that

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0$$

for all integers $n \geq 0$.

(b) More generally, show that

$$\lim_{x \to \infty} \frac{x^t}{e^x} = 0$$

for all *real* $t \geq 0$.

[*Hint*: For any real number t, there is an integer $n > t$.]

Q3. **[12 marks]** Consider the function $f(x) = e^{-x}\ln(x)$ for $1 \le x < \infty$.

(a) Using L'Hôpital's rule, show that

$$\lim_{x\to\infty} e^{-x}\ln(x) = 0.$$

(b) Use Maple to graph the functions $y = \frac{1}{x}$ and $y = \ln(x)$ for $x \ge 1$.

Use these graphs to explain why the function $g(x) = \frac{1}{x} - \ln(x)$ has only one zero in the interval $[1, \infty)$.

(c) Let a be this zero of $g(x)$. What is the sign of $g(x)$ in the interval $[1, a)$, and in the interval (a, ∞)?

(d) We now want to calculate a using Newton's method, starting at $x_0 = 1$.

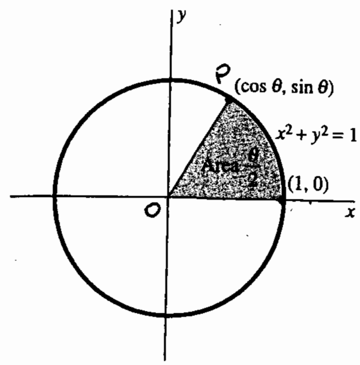
(i) Write down the form the iteration formula takes for this specific function g.

(ii) Suppose that the iteration process stops when $|x_n - x_{n-1}| < 10^{-2}$. Using Maple to do your calculations, record the values of x_n and $g(x_n)$ which you obtain.

(iii) Check your estimate for a, obtained in [(*ii*)] above, using the Maple fsolve command.

(e) Find $f'(x)$, and using the results of parts (a)–(d), sketch the graph of $y = f(x)$, for $1 \le x < \infty$.

Q4. [14 marks] The trigonometric functions are based on properties of the circle. For example, a point on the circle $x^2 + y^2 = 1$ has coordinates $(\cos\theta, \sin\theta)$ where θ is the angle that the line joining the origin to the point, makes with the positive x−axis (see the diagram below). Notice that θ is twice the area of the shaded *circular sector.*



In a similar way, the *hyperbola* $x^2 - y^2 = 1$ leads to the study of *hyperbolic* functions.

The *hyperbolic cosine* and *hyperbolic sine* of the real number θ are denoted by $\cosh\theta$ and $\sinh\theta$ and are defined to be

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2} \quad \text{and} \quad \sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$$

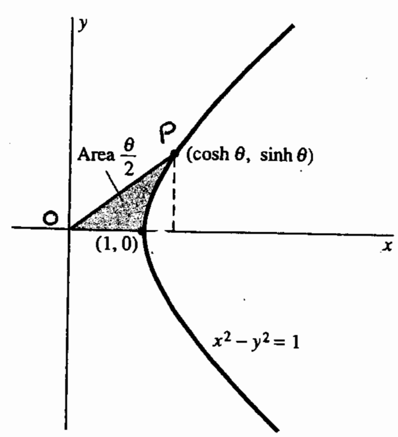
(a) Show that

(i) $\cosh^2\theta - \sinh^2\theta = 1$

(ii) $\dfrac{d}{d\theta}\sinh\theta = \cosh\theta$

(iii) $\dfrac{d}{d\theta}\cosh\theta = \sinh\theta$

Now consider the diagram below.



(b) Show that the area of the shaded *hyperbolic sector*, $A(\theta)$ can be expressed as

$$A(\theta) = \frac{1}{2}\cosh\theta\sinh\theta - \int_1^{\cosh\theta} \sqrt{x^2 - 1}\,dx$$

(c) Show that $A'(\theta) = \dfrac{1}{2}$

(You will need to apply the second fundamental theoremn of calculus, and the chain rule here.)

(d) Deduce from (c) that $A(\theta) = \dfrac{\theta}{2}$.