Jack Huizenga's Solution

Let $0 < \delta < \pi$, and define $f_\delta : R \to R$ by $f_\delta(x) = 1$ if $|x| \le \delta$, $f_\delta(x) = 0$ if $\delta < |x| \le \pi$, and $f_\delta(x + 2\pi) = f_\delta(x)$ for every x. Then f_δ is periodic with period 2π, and is the pointwise limit of its Fourier series everywhere f is continuous, so for $x \neq \pm\delta$ we have

$$f_\delta(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f_\delta(t) \cos kt \, dt$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f_\delta(t) \sin kt \, dt.$$

Since the form of f_δ is particularly nice, these integrals are very easy to calculate: for $k > 0$, we have

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f_\delta(t) \cos kt \, dt = \frac{1}{\pi} \int_{-\delta}^{\delta} \cos kt \, dt = \left.\frac{\sin kt}{k\pi}\right|_{-\delta}^{\delta} = \frac{2 \sin k\delta}{k\pi}$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f_\delta(t) \sin kt \, dt = \frac{1}{\pi} \int_{-\delta}^{\delta} \sin kt \, dt = 0.$$

Also,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f_\delta(t) \, dt = \frac{2\delta}{\pi}.$$

Therefore

$$f_\delta(x) = \frac{\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2 \sin n\delta}{n\pi} \cos(nx).$$

In particular,

$$1 = f_\delta(0) = \frac{\delta}{\pi} + \sum_{n=1}^{\infty} \frac{2 \sin n\delta}{n\pi},$$

so

$$\sum_{n=1}^{\infty} \frac{\sin n\delta}{n} = \frac{\pi - \delta}{2}.$$

With this identity, we are ready to calculate the integral at hand. We note briefly that $\int_0^\infty \frac{\sin x}{x} \, dx$ clearly converges since the integrals over each interval $[n, n+1]$ form an alternating sequence whose terms successively decrease in absolute value and converge to zero. Then since $\frac{\sin x}{x}$ is continuous on every interval $[0, N]$, we have that

$$\int_0^\infty \frac{\sin x}{x} \, dx = \lim_{\delta \to 0} \sum_{n=1}^{\infty} \frac{\sin n\delta}{n\delta} \cdot \delta = \lim_{\delta \to 0} \sum_{n=1}^{\infty} \frac{\sin n\delta}{n} = \lim_{\delta \to 0} \frac{\pi - \delta}{2} = \frac{\pi}{2}.$$