Let p be an offspring distribution for a branching process such that $p(0) > 0$ and $\mu \geq 1$. Let φ be the generating function for p. Let X_n denote the number of individuals in the nth generation and assume $X_0 = 1$.

(a) *If $\mu = 1$ and $\sigma^2 < \infty$, then there exist c_1, c_2 such that for $n \geq 1$, $c_1/n \leq P\{X_n \neq 0\} \leq c_2/n$.*

Let $b_n = 1 - a_n = P\{X_n \neq 0\}$, so $b_{n+1} = 1 - \varphi(1 - b_n)$.

We have $a_1 = \varphi(0) > 0$, $\varphi(1) = 1$, and $\varphi'(1) = \mu = 1$. Because $\mu = 1$, we have $p(0) < 1$, so $b_1 = 1 - \varphi(0) > 0$. Also, because $p(0) > 0$ and $\mu = 1$, we must have $p(k) > 0$ for some $k \geq 2$, so $\varphi''(t) > 0$ for $t > 0$.

Since $\sigma^2 < \infty$, φ'' is continuous on the compact interval $[a_1, 1]$, and we may pick $0 < r_1 < r_2$ such that

$$r_1 < \varphi''(t) < r_2 \text{ for } t \in [a_1, 1].$$

Because $b_n \to 0$, we may pick $N > 2$ such that $b_n < 1/r_2 < 1/r_1$ for $n \geq N$. Now we may pick c_2 such that $N/r_1 > c_2 > N b_N$ and $c_2 > 2/r_1$. Pick c_1 less than both b_N/N and $1/r_2$.

We prove $c_1/n < b_n < c_2/n$ for $n \geq N$. The case $n = N$ is true by assumption.

By Taylor's Theorem, for some $t \in [a_n, 1] \subset [a_1, 1]$,

$$b_{n+1} = 1 - \varphi(1 - b_n) = 1 - \left(\varphi(1) - \varphi'(1)b_n + \varphi''(t)\frac{b_n^2}{2}\right) = b_n - \frac{\varphi''(t)}{2}b_n^2.$$

Now the function $f_1(t) = t - r_1 t^2/2$ is increasing for $t \in [0, 1/r_1] \supset [0, c_2/N]$. It follows that

$$b_{n+1} < f_1(b_n) < f(c_2/n) = \frac{c_2}{n}\left(1 - \frac{r_1}{2}\frac{c_2}{n}\right) < \frac{c_2}{n}\left(1 - \frac{1}{n}\right) = c_2\frac{n-1}{n^2} < \frac{c_2}{n+1}.$$

Similarly, $f_2(t) = t - r_2 t^2/2$ is increasing for $t \in [0, 1/r_2] \supset [0, b_n]$, so

$$b_{n+1} > f_2(b_n) > f_2(c_1/n) = \frac{c_1}{n}\left(1 - \frac{r_2}{2}\frac{c_1}{n}\right) > \frac{c_1}{n}\left(1 - \frac{1}{2n}\right) = c_1\frac{2n-1}{2n^2} > \frac{c_1}{n+1}.$$

Since $c_1/n < b_n < c_2/n$ for $n \geq N$, letting $C_2 = \max(c_2, \max_{n=1}^N n b_n)$, $C_1 = \min(c_1, \min_{n=1}^N n b_n)$, shows $C_1/n \leq b_n \leq C_2/n$ for all $n \geq 1$.

(b) *If $\mu > 1$ and $a < 1$ is the extinction probability, then $\varphi'(a) < 1$.*

$$1 - a = 1 - \varphi(a) = \int_a^1 \varphi'(t)dt > \int_a^1 \varphi'(a)dt = (1 - a)\varphi'(a)$$

because $\varphi'(t)$ is (strictly) increasing, and it follows $\varphi'(a) < 1$.

(c) *If $\mu > 1$, there exist $c, b > 0$ such that for all $n > 1$, $P(\text{extinction}|X_n \neq 0) \leq ce^{-bn}$.*

$$\begin{aligned} P(\text{extinction}) &= P(\text{extinction}|X_n = 0)P(X_n = 0) + P(\text{extinction}|X_n \neq 0)P(X_n \neq 0) \\ a &= a_n + P(\text{extinction}|X_n \neq 0)(1 - a_n) \end{aligned}$$

Let $\delta_n = a - a_n$. By Taylor's Theorem, for some $t \in [a_n, a]$,

$$a_{n+1} = \varphi(a_n) = \varphi(a - \delta_n) = \varphi(a) - \varphi'(a)\delta_n + \varphi''(t)\delta_n^2/2$$

$$\delta_{n+1} = a - a_{n+1} = \varphi(a) - a_{n+1} = \varphi'(a)\delta_n - \varphi''(t)\delta_n^2/2 \leq \varphi'(a)\delta_n.$$

It follows $\delta_n \leq \varphi'(a)^n \delta_0 = \varphi'(a)^n a$. Since also $1 - a_n \geq 1 - a$,

$$P(\text{extinction}|X_n \neq 0) = \frac{a - a_n}{1 - a_n} \leq \varphi'(a)^n \frac{a}{1 - a} = ce^{-bn},$$

with $b = -\log \varphi'(a) > 0$, $c = a/(1 - a)$.