Finding the value of $\tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right)$

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Recall the identities

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}.$$

Using these, and writing $\omega = e^{2\pi i/11}$ for an eleventh root of unity, we obtain

$$\begin{aligned}
\tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right) &= \frac{\omega^{3/2} - \omega^{-3/2}}{i\left(\omega^{3/2} + \omega^{-3/2}\right)} + \frac{2\left(\omega - \omega^{-1}\right)}{i} \\
&= -i\left(\frac{\omega^3 - 1}{\omega^3 + 1} + 2\frac{\omega^2 - 1}{\omega}\right).
\end{aligned}$$

Since $3\pi/11$ and $2\pi/11$ are both in the first quadrant, their tangents and sines are positive. Hence, the entire sum is positive and it may be written as the square root of its square:

$$\begin{aligned}
\tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right) &= \sqrt{\left(-i\left(\frac{\omega^3 - 1}{\omega^3 + 1} + 2\frac{\omega^2 - 1}{\omega}\right)\right)^2} \\
&= \sqrt{-\left(\frac{\omega^6 - 2\omega^3 + 1}{\omega^6 + 2\omega^3 + 1} + 4\frac{\omega^4 - 2\omega^2 + 1}{\omega^2} + 4\frac{\omega^5 - \omega^3 - \omega^2 + 1}{\omega^4 + \omega}\right)} \\
&= \sqrt{-\frac{\left(\begin{array}{c} \omega^8 - 2\omega^5 + \omega^2 + \ldots \\ \ldots + 4\omega^{10} + 8\omega^7 + 4\omega^4 - 8\omega^8 - 16\omega^5 - 8\omega^2 + 4\omega^6 + 8\omega^3 + 4 + \ldots \\ \ldots + 4\omega^9 + 4\omega^6 - 4\omega^7 - 4\omega^4 - 4\omega^6 - 4\omega^3 + 4\omega^4 + 4\omega \end{array}\right)}{\omega^8 + 2\omega^5 + \omega^2}} \\
&= \sqrt{-\frac{4\omega^{10} + 4\omega^9 - 7\omega^8 + 4\omega^7 + 4\omega^6 - 18\omega^5 + 4\omega^4 + 4\omega^3 - 7\omega^2 + 4\omega + 4}{\omega^8 + 2\omega^5 + \omega^2}} \\
&= \sqrt{-\frac{\left(4\sum_{k=0}^{10}\omega^k\right) - 11\omega^8 - 22\omega^5 - 11\omega^2}{\omega^8 + 2\omega^5 + \omega^2}} \\
&= \sqrt{11 - \frac{4\sum_{k=0}^{10}\omega^k}{\omega^8 + 2\omega^5 + \omega^2}}.
\end{aligned}$$

But, since we defined ω to be an eleventh root of unity, the numbers $\omega^0, \omega^1, \ldots, \omega^{10}$ are just all distinct eleventh roots of unity! Therefore, they must sum to zero[1] and we have

$$\tan\left(\frac{3\pi}{11}\right) + 4\sin\left(\frac{2\pi}{11}\right) = \sqrt{11}.$$

[1] The summands are the roots of the polynomial $z^{11} - 1$. Their sum must be equal to the negative of the coefficient of z^{10} in this polynomial, which is zero.