Problem: Let $x_1, x_2, x_3, \dots$ be positive real numbers. For positive integer n, denote $A_n = \frac{x_1+\ldots+x_n}{n}$. Use the inequality $(a+b)^n \geq a^n + na^{n-1}b$ to prove the following statement:

$$\forall n \in \mathbb{N} : A_{n+1}^{n+1} \geq A_n^n \cdot x_{n+1}$$

Attempted Solution: Let $a = x_1 + \ldots + x_n$ and $b = x_{n+1}$. Then, applying the inequality $(a+b)^n \geq a^n + na^{n-1}b$:

$$\begin{aligned}
[(x_1+\ldots+x_n)+x_{n+1}]^{n+1} &\geq (x_1+\ldots+x_n)^{n+1} + (n+1)(x_1+\ldots+x_n)^n x_{n+1} \\
\left(\frac{1}{n+1}\right)^{n+1}[(x_1+\ldots+x_n)+x_{n+1}]^{n+1} &\geq \left(\frac{1}{n+1}\right)^{n+1}\left[(x_1+\ldots+x_n)^{n+1} + (n+1)(x_1+\ldots+x_n)^n x_{n+1}\right] \\
A_{n+1}^{n+1} &\geq \left(\frac{1}{n+1}\right)^{n+1}\left[(x_1+\ldots+x_n)^{n+1} + (n+1)(x_1+\ldots+x_n)^n x_{n+1}\right] \\
\left(\frac{1}{n}\right)^n A_{n+1}^{n+1} &\geq \left(\frac{1}{n}\right)^n\left(\frac{1}{n+1}\right)^{n+1}\left[(x_1+\ldots+x_n)^{n+1} + (n+1)(x_1+\ldots+x_n)^n x_{n+1}\right] \\
\left(\frac{1}{n}\right)^n A_{n+1}^{n+1} &\geq \left(\frac{1}{n+1}\right)^{n+1}\left[\frac{(x_1+\ldots+x_n)^{n+1} + (n+1)(x_1+\ldots+x_n)^n x_{n+1}}{n^n}\right] \\
\left(\frac{1}{n}\right)^n A_{n+1}^{n+1} &\geq \left(\frac{1}{n+1}\right)^{n+1}\left[\frac{(x_1+\ldots+x_n)^n(x_1+\ldots+x_n)}{n^n} + \frac{(n+1)(x_1+\ldots+x_n)^n x_{n+1}}{n^n}\right] \\
\left(\frac{1}{n}\right)^n A_{n+1}^{n+1} &\geq \left(\frac{1}{n+1}\right)^{n+1}\left[A_n^n(x_1+\ldots+x_n) + (n+1)A_n^n x_{n+1}\right]
\end{aligned}$$

Since $\forall i \in \mathbb{N} : x_i > 0$, we can eliminate $A_n^n(x_1+\ldots+x_n)$ from the right-hand side of the above inequality, thereby loosening the inequality:

$$\begin{aligned}
\left(\frac{1}{n}\right)^n A_{n+1}^{n+1} &\geq \left(\frac{1}{n+1}\right)^{n+1}(n+1)A_n^n x_{n+1} \\
\left(\frac{1}{n}\right)^n A_{n+1}^{n+1} &\geq \left(\frac{1}{n+1}\right)^{n} A_n^n x_{n+1} \\
A_{n+1}^{n+1} &\geq \left(\frac{n}{n+1}\right)^{n} A_n^n \cdot x_{n+1}
\end{aligned}$$

This falls short of proving the desired inequality. The earlier loosening is probably where I went astray, but I tried playing with that term I threw out and didn't get very far. - WW